Fourier Series and Fourier Transform

One can superimpose plane waves with different $k$ values to construct a wave packet

$$
\Psi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, \Psi(k) \exp(ikx). \tag{1}
$$

Here $\Psi(k)$ is the amplitude of the plane wave with wave number $k$. The above equation describes a Fourier transformation, and $\Psi(k)$ is called the Fourier transform of the function $\Psi(x)$. Note that $\Psi(k)$ and $\Psi(x)$ are two different functions. They are denoted by the same letter $\Psi$ just because they are closely related via the Fourier transform.

The Fourier transform is a generalization of a Fourier series, which can be used to describe periodic functions. Let us assume for a moment that $\Psi(x)$ is defined in the interval $[-L/2, L/2]$ and that it is periodically extended outside that interval. Then we can write it as a Fourier series

$$
\Psi(x) = \frac{1}{L} \sum_{k} \Psi_k \exp(ikx). \tag{2}
$$

Here $\Psi_k$ is the amplitude of the mode with wave number $k$. Due to the periodicity requirement the wave lengths of the various modes must now be an integer fraction of $L$, and therefore the $k$ values are restricted to

$$
k = \frac{2\pi}{L} m, \ m \in \mathbb{Z}. \tag{3}
$$

The separation of two modes $dk = 2\pi/L$ becomes infinitesimal in the large volume limit $L \rightarrow \infty$. In this limit the sum over modes turns into an integral

$$
\Psi(x) \rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, \Psi_k \exp(ikx). \tag{4}
$$

When we identify $\Psi(k) = \Psi_k$ we recover the above expression for a Fourier transform. We can use our knowledge of the Fourier series to derive further relations for Fourier transforms. For example, we know that the amplitudes in a Fourier series are obtained as

$$
\Psi_k = \int_{-L/2}^{L/2} dx \ \Psi(x) \exp(-ikx). \tag{5}
$$
Let us convince ourselves that this is indeed correct by inserting eq.(2) in this expression

\[
\Psi_k = \int_{-L/2}^{L/2} dx \frac{1}{L} \sum_{k'} \Psi_{k'} \exp(i k' x) \exp(-i k x)
\]

\[
= \sum_{k'} \Psi_{k'} \frac{1}{L} \int_{-L/2}^{L/2} dx \exp(i(k' - k)x) = \sum_{k'} \Psi_{k'} \delta_{k,k'} .
\]

(6)

Here we have identified the Kronecker \( \delta \)-function

\[
\delta_{k',k} = \frac{1}{L} \int_{-L/2}^{L/2} dx \exp(i(k' - k)x),
\]

(7)

which is 1 for \( k' = k \) and 0 otherwise. Then indeed the above sum returns \( \Psi_k \). Let us now take the \( L \to \infty \) limit of eq.(5) in which we get

\[
\Psi_k \to \int_{-\infty}^{\infty} dx \; \Psi(x) \exp(-i k x).
\]

(8)

Again identifying \( \Psi(k) = \Psi_k \) we find the expression for an inverse Fourier transform

\[
\Psi(k) = \int_{-\infty}^{\infty} dx \; \Psi(x) \exp(-i k x).
\]

(9)

Let us check the validity of this expression in the same way we just checked eq.(5). We simply insert eq.(1) into eq.(9)

\[
\Psi(k) = \int_{-\infty}^{\infty} dx \frac{1}{2 \pi} \int_{-\infty}^{\infty} dk' \Psi(k') \exp(i k' x) \exp(-i k x)
\]

\[
= \int_{-\infty}^{\infty} dk' \Psi(k') \frac{1}{2 \pi} \int_{-\infty}^{\infty} dx \exp(i(k' - k)x)
\]

\[
= \int_{-\infty}^{\infty} dk' \Psi(k') \delta(k' - k).
\]

(10)

Here we have identified

\[
\delta(k' - k) = \frac{1}{2 \pi} \int_{-\infty}^{\infty} dx \exp(i(k' - k)x),
\]

(11)

which is known as the Dirac \( \delta \)-function. It is the continuum analog of the discrete Kronecker \( \delta \)-function. The Dirac \( \delta \)-function has the remarkable property that

\[
\int_{-\infty}^{\infty} dk' \; \Psi(k') \delta(k' - k) = \Psi(k).
\]

(12)

This already follows from the fact that the limit \( L \to \infty \) led to eq.(9).