Microwave Resonator

Laboratory Course Modern Physics
Institute of Applied Physics, University of Bern

Version: 21.10.2020

1 Introduction

The term microwaves refers to alternating electromagnetic signals with frequencies between 300 MHz and 300 GHz, or wavelengths between 1 m and 1 mm. Because of the longer wavelengths with respect to the visible and infrared parts of the electromagnetic spectrum, microwaves exhibit unique properties. For instance, microwave radiation can penetrate through cloud cover, haze, dust, and even rain as they are less susceptible to atmospheric scattering. In addition, various molecules and atoms exhibit spectral lines in the microwave range. Microwave applications are found in fields such as

- Wireless communication: mobile networks, WLAN, satellite communication, satellite navigation systems (GPS, Galileo, ...),
- Radiometry: environmental remote sensing (atmosphere and surface of the Earth and other planets), radio astronomy (molecular gas clouds, quasars, the Sun), security (concealed threat detection),
- Radar systems for commercial, scientific and military purposes.

The Microwave Physics group at the University of Bern is specialized in microwave radiometry and remote sensing of the Earth's atmosphere and surface. The group designs, builds, and operates passive microwave radiometers for applications such as wind, water vapour and ozone monitoring. It also actively participates in the development of space based radiometers for the missions of the European Space Agency (ESA). Characterization of electric and magnetic material parameters is a task often related to the instrument development and analysis of observational data. The permittivity and permeability of a microwave absorbing material need to be accurately known in order to design a high-performance blackbody calibration target for a radiometer. This is also the case when designing low-loss dielectric vacuum windows for a cryostat of a cryogenically cooled radiometer: especially the radio astronomical radiometers are cooled in a cryostat to obtain better sensitivity, and the vacuum window allows the electromagnetic signal from the observed target to enter it. Furthermore, characterization of the material parameters of snow, ice and soil samples allows one to derive the thicknesses of surface layers from remote sensing data obtained from an airplane or a satellite.

This laboratory exercise demonstrates how the permittivity of a dielectric sample can be measured using a cylindrical waveguide cavity resonator. In addition to learning to measure the permittivities of different materials, the goal of the exercise is to understand the position of microwaves in the electromagnetic spectrum, to review the meaning of permittivity, to become more familiar with waveguides, waveguide modes, and resonators, and to gain first experience with a vector network analyzer (VNA) that is an essential tool in microwave measurements.

2 Theory of a Cylindrical Cavity Resonator

For electromagnetic oscillations with wavelengths between a few decimeter and a few millimeter, resonant circuits are composed of cavity resonators which exhibit highly conducting boundaries. Depending on the shape and size of the resonator, resonances occur at certain frequencies in different types of field patterns called modes (see Appendix A and B). Assuming a cylindrical cavity with its main axis oriented in z-direction, one distinguishes between transverse electric (TE) modes, where $E_z = 0$, and transversal magnetic (TM) modes, where $H_z = 0$.

For the mathematical derivation of TE and TM modes in cylindrical cavities, the common procedure taken from [1–5] is used, where Maxwell's equations in frequency domain are applied:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon}, \\ \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{E} &= -i\omega \mathbf{B}, \quad \text{and} \\ \nabla \times \mathbf{H} &= \mathbf{j} + i\omega\epsilon \mathbf{E}\,. \end{aligned} \tag{1}$$

The time dependence is assumed to be $e^{i\omega t}$. Hence, the derivation with respect to time is considered by the factor $i\omega$ in the equations above. As no free charges nor conducting materials exist inside the cavity, Maxwell's equation reduce to

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0, \\ \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{E} &= -i\omega\mu \mathbf{H}, \quad \text{and} \\ \nabla \times \mathbf{H} &= i\omega\epsilon \mathbf{E}\,. \end{aligned} \tag{2}$$

The application of the rotation operator as well as a substitution leads to the homogeneous Helmholtz equation

$$\begin{aligned} \nabla^2 \mathbf{E} &= -\mu\epsilon\omega^2 \mathbf{E} \quad \text{and} \\ \nabla^2 \mathbf{H} &= -\mu\epsilon\omega^2 \mathbf{H}\,. \end{aligned} \tag{3}$$

For waves inside a waveguide which travel in z-direction, it is useful to split the wave into a transverse and longitudinal part as:

$$\nabla^2 \mathbf{E} = \nabla_t^2 \mathbf{E} + \frac{\partial^2 \mathbf{E}}{\partial z^2} \quad . \tag{4}$$

In direction of propagation (z-direction), E_z has the form

$$E_z = E_t \cdot e^{\pm\gamma z} \tag{5}$$

and it has to fulfill

$$\nabla_t^2 E_z = -(\gamma^2 + k^2)E_z = -p^2 E_z \tag{6}$$

with

$$\nabla_t^2 = \nabla^2 - \frac{\partial^2}{\partial z^2}\,. \tag{7}$$

The analog holds for the magnetic field in (3).

For the transverse part and in case of cylindrical symmetry, the following ansatz is appropriate:

$$E_t = R(r)\theta(\phi)\,. \tag{8}$$

Using the Laplace operator for cylindrical coordinates, (6) may be written as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}R\theta = -p^2 R\theta\,. \tag{9}$$

or

$$\frac{1}{R}\left(r^2\frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r}\right) + r^2p^2 = \frac{-1}{\theta}\frac{\partial^2\theta}{\partial\phi^2} = \ell^2\,. \tag{10}$$

For a non-trivial solution, the r-depending as well as the ϕ-depending terms have to equal a constant ℓ^2. The two independent solutions of the r- and ϕ-dependent portions are

$$\frac{\partial^2\theta}{\partial\phi^2} = -\ell^2\theta \tag{11}$$

and

$$r^2\frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r} + R\left(p^2r^2 - \ell^2\right) = 0\,, \tag{12}$$

respectively. The solution of (11) is

$$\theta(\phi) = A\sin(\ell\phi) + B\cos(\ell\phi)\,, \tag{13}$$

and the solution of the Bessel differential equation (12) is given by

$$R(r) = CJ_\ell(pr) + DN_\ell(pr)\,, \tag{14}$$

where $J_\ell(pr)$ is a Bessel function and $N_\ell(pr)$ is a Neumann function. As the Neumann functions exhibit a singularity at $r = 0$, they do not represent a physically meaningful solution. Hence, only the Bessel function J_ℓ is considered. The Bessel functions of different orders and their derivatives are depicted in Appendix C. Using these functions, the following relation holds for E_z as well as for H_z:

$$E_z = CJ_\ell(pr)(A\sin\ell\phi + B\cos\ell\phi)e^{\pm\gamma z}\,. \tag{15}$$

Applying the rotation of the E- and H-field, respectively, the transverse components can be calculated from the longitudinal component. The fields in cylindrical coordinates are

$$\begin{aligned}
E_r &= \frac{1}{p^2}\left[\gamma\frac{\partial E_z}{\partial r} - \frac{i\omega\mu}{r}\frac{\partial H_z}{\partial\phi}\right]\,,\\
E_\phi &= \frac{1}{p^2}\left[\frac{\gamma}{r}\frac{\partial E_z}{\partial\phi} + i\omega\mu\frac{\partial H_z}{\partial r}\right]\,,\\
H_r &= \frac{1}{p^2}\left[\frac{i\omega\epsilon}{r}\frac{\partial E_z}{\partial\phi} + \gamma\frac{\partial H_z}{\partial r}\right]\,,\quad\text{and}\\
H_\phi &= \frac{1}{p^2}\left[-i\omega\epsilon\frac{\partial E_z}{\partial r} + \frac{\gamma}{r}\frac{\partial H_z}{\partial\phi}\right]\,.
\end{aligned} \tag{16}$$

Analogously to the cylindrical case, the field equations for a rectangular cavity can be derived following the procedure given above using the Cartesian coordinates.

2.1 TM Mode

For the transverse magnetic mode the z-component of the magnetic field is zero, i. e. $H_z = 0$. From equation (16) follows that the components E_r, E_ϕ, H_r und H_ϕ do only depend on E_z. The longitudinal component of the E-field vanishes at the cavity walls. Hence, the boundary boundary condition

$$E_z(r = R) = 0 \tag{17}$$

holds, where R is the radius of the cylindrical cavity. The electric field is zero if

$$J_\ell(p_{\ell m}R) = J_\ell(Y_{\ell m}) = 0\,. \tag{18}$$

The m-th zero of the Bessel function is denoted by $Y_{\ell m}$. Values for the zeros of Bessel functions are found in the literature, e.g. [5]. From this, the following TM$_{\ell m}$ modes result:

$$E_z = C_\ell J_\ell\left(Y_{\ell m}\frac{r}{R}\right)\cos(\ell\phi)\, e^{\pm\gamma z}\,. \tag{19}$$

The transversal components are

$$\begin{aligned}
E_r &= -\frac{\gamma R}{Y_{\ell m}} C_\ell J'_\ell\left(Y_{\ell m}\frac{r}{R}\right)\cos(\ell\phi)e^{\pm\gamma z}\,,\\
E_\phi &= \frac{\gamma\ell R^2}{Y^2_{\ell m} r} C_\ell J_\ell\left(Y_{\ell m}\frac{r}{R}\right)\sin(\ell\phi)e^{\pm\gamma z}\,,\\
H_r &= \frac{i\omega\epsilon}{\gamma}E_\phi\,, \quad \text{and}\\
H_\phi &= \frac{-i\omega\epsilon}{\gamma}E_r\,.
\end{aligned} \tag{20}$$

If the cavity is closed (terminated with a conducting wall at each end), the general solution consist of a forward and backward traveling wave:

$$E_z = \left[Ae^{\gamma z} + Be^{-\gamma z}\right] J_\ell\left(Y_{\ell m}\frac{r}{R}\right)\cos(\ell\phi)\,. \tag{21}$$

For lossless media $\gamma = i\beta$. As the transversal electric field has to vanish at $z = 0$ und $z = L$, the constants A, B, and γ are chosen such that a $\sin\frac{n\pi z}{L}$ function in z-direction results. With help of the relations $\beta = \frac{n\pi}{L}$ and $k^2_{\ell mn} = p^2_{\ell m} + \beta^2$, the resonance frequency of a lossless cylindrical cavity can be determined by:

$$f_{\ell mn} = \frac{c}{2}\sqrt{\left(\frac{Y_{\ell m}}{\pi R}\right)^2 + \left(\frac{n}{L}\right)^2}\,. \tag{22}$$

Please note: The z-dependence of E_r and E_ϕ is $\sin\gamma z$ as shown above. As E_t results from the derivation of E_z with respect to z, the component E_z includes a $\cos\beta z$ term. The maxima of the magnetic field are $\frac{\pi}{2}$ phase shifted and have a $\cos\beta z$ dependence.

2.2 TE Mode

For this mode the longitudinal electric field is zero $E_z = 0$. The boundary conditions for the TE mode is:

$$\left.\frac{\partial H_z}{\partial r}\right|_{r=R} = 0\,, \tag{23}$$

which leads to

$$J'_\ell(p_{\ell m}R) = J'_\ell(Y'_{\ell m}) = 0\,, \tag{24}$$

analogously to the TM mode. This condition is required to make the tangential E_θ vanish at the walls of the resonator. The field components are

$$\begin{aligned} H_z &= CJ_\ell\left(Y'_{\ell m}\frac{r}{R}\right)\cos(\ell\phi)e^{\pm\gamma z}\,, \\ E_r &= \frac{i\omega\mu\ell R^2}{Y'^2_{\ell m}r}CJ_\ell\left(Y'_{\ell m}\frac{r}{R}\right)\sin(\ell\phi)e^{\pm\gamma z}\,, \\ E_\phi &= \frac{i\omega\mu R}{Y'_{\ell m}}CJ'_\ell\left(Y'_{\ell m}\frac{r}{R}\right)\cos(\ell\phi)e^{\pm\gamma z}\,, \\ H_r &= \frac{i\gamma}{\omega\mu}E_\phi\,, \quad \text{and} \\ H_\phi &= \frac{i\gamma}{\omega\mu}E_r\,. \end{aligned} \tag{25}$$

The resonance frequency can be determined with (22), but here, the zero of the first derivation of the Bessel function $Y'_{\ell m}$ has to be applied. As the lines of the magnetic field have to be closed inside the cavity, H_z possesses a $\sin\beta z$ dependence. The z-dependence of E_ϕ, E_r, H_ϕ and H_r is the same as for the TM mode.

3 Permittivity Measurement of Dielectric Slabs

The measurements in this experiment are carried out by means of a vector network analyzer (VNA) connected to a cylindrical cavity with an adjustable length. The measurement setup is illustrated in Fig. 1. Only the $\text{TE}_{\ell mn}$ modes are considered in this configuration. We measure the magnitude of the transmission coefficient in the decibel scale as a function of frequency between the two ports of the cavity resonator. Fig. 2 shows an example measurement of the TE_{211} mode resonance at a frequency of 10.5 GHz in an empty cylindrical cavity with a radius R = 20 mm and length L = 20.09 mm.

The resonance frequency of a resonant mode changes when a dielectric slab is inserted into the cavity. The dielectric properties of the inserted slab can be extracted from this frequency shift. Alternatively, we may adjust the cavity length in order to keep the resonance frequency unchanged, and use the length difference information to determine the dielectric properties. The cavity resonator with a dielectric slab are illustrated in Fig. 3. In the following, two methods for computing the permittivity of the slab are introduced: the perturbation theory as well as the exact solution of the modified boundary value problem.

Figure 1: Experimental setup of the cylindrical resonator connected to a vector network analyzer.

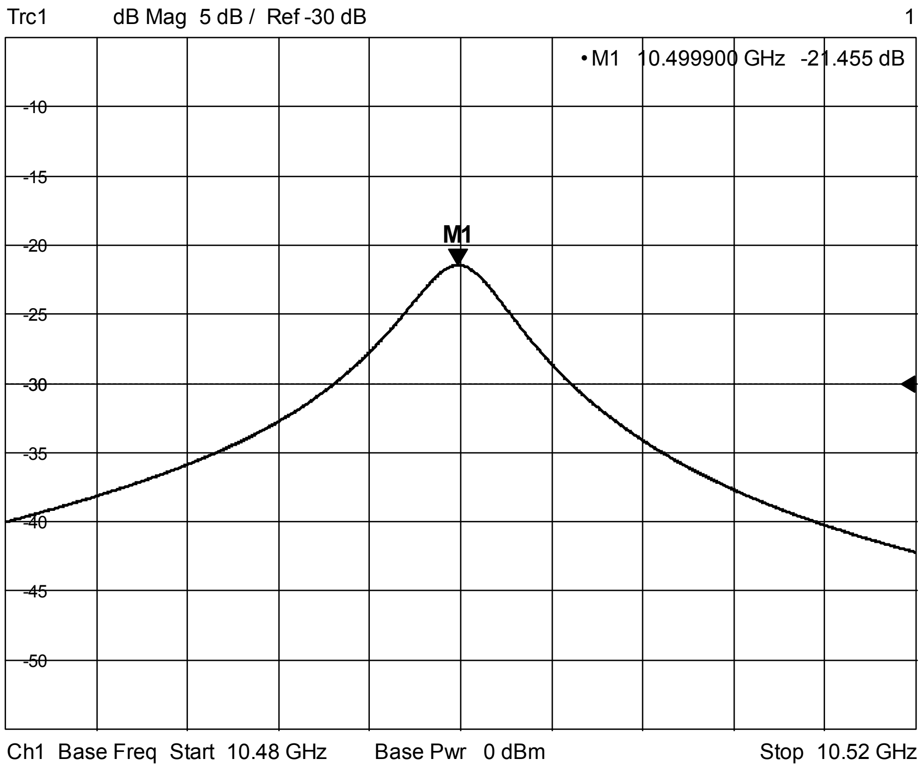


Figure 2: Microwave transmission of a cylindrical resonator using the dB-scale (5 dB per horizontal grid line) in the frequency range of the TE_{211} mode.

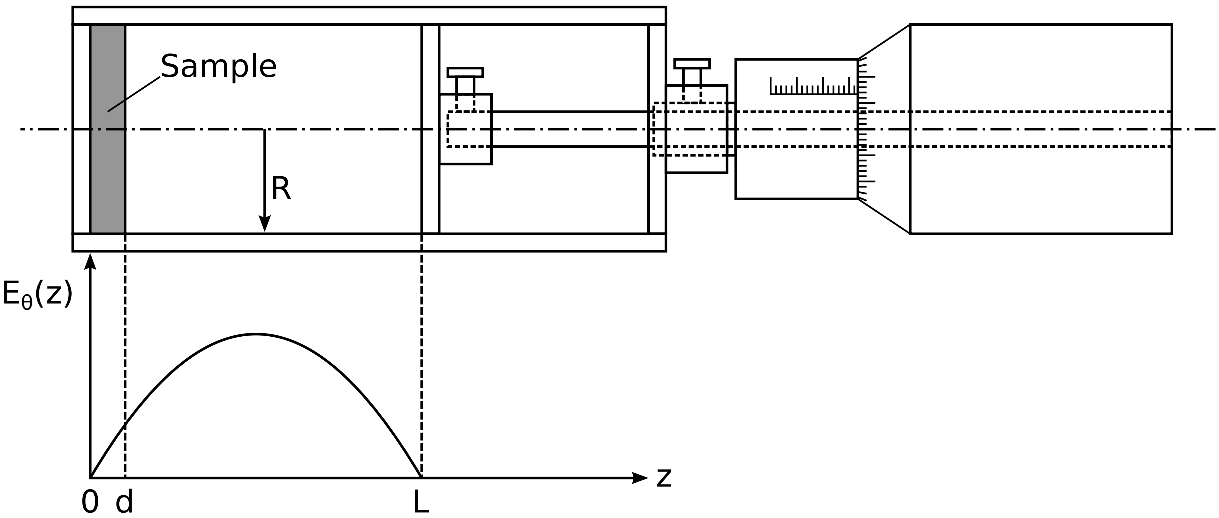


Figure 3: Cylindrical resonator with varying length L and associated magnitude of the electric field.

3.1 Perturbation Theory

The perturbation theory represents an approximation to the real solution. By changing the permittivity of the (non-magnetic) filling material of the cavity from ϵ by $\Delta\epsilon$, the resonant frequency is changed according to the equation

$$\frac{f-f_0}{f} = \frac{\Delta f}{f} = -\frac{\int\limits_V \Delta\epsilon E \cdot E_0^* \, dV}{\int\limits_V \epsilon E \cdot E_0^* + \mu H \cdot H_0^* \, dV}, \tag{26}$$

where E_0 is the field of the empty cavity and E is the electric field of the filled cavity [6]. When the amount of inserted material is small (i.e. the dielectric slab is thin), the perturbed field can be approximated by the original field. Moreover, in case of resonances, the magnetic and electric energy are equal. Hence, (26) may be written as

$$\frac{f-f_0}{f} = \frac{\Delta f}{f} \approx -\frac{\int\limits_V \Delta\epsilon |E_0|^2 \, dV}{2\int\limits_V \epsilon |E_0|^2 \, dV} = -\frac{\int\limits_{V_1} \Delta\epsilon |E_0|^2 \, dV_1 + \overbrace{\int\limits_{V_2} \Delta\epsilon |E_0|^2 \, dV_2}^{=0}}{2\int\limits_V \epsilon |E_0|^2 \, dV}, \tag{27}$$

where the total volume V is split into portions V_1 and V_2 corresponding to the dielectric slab and the remaining cavity, respectively. As the material changes only in the volume V_1, the second integral associated with V_2 is zero.

3.2 Exact Solution

The propagation constant of a $\mathrm{TE}_{\ell mn}$ mode in the z-direction in an empty resonator with a length L_0 and a radius R can be obtained from (22):

$$\beta_0 = \sqrt{\mu_0\epsilon_0\omega^2 - k_c^2} = \frac{n\pi}{L_0}, \tag{28}$$

where the cutoff constant is

$$k_c = \frac{Y'_{\ell m}}{R}. \tag{29}$$

The same $\mathrm{TE}_{\ell mn}$ mode can also be excited in the dielectric slab with a thickness d. The propagation constant in this case is

$$\beta_1 = \sqrt{\mu_0\epsilon_0\epsilon_r\omega^2 - k_c^2}, \tag{30}$$

where ϵ_r is the relative permittivity of the sample. The boundary conditions at the interface between the sample and the empty cavity at $z = d$ require that the tangential components of $\mathbf{E}$ and $\mathbf{H}$ are continuous:

$$\beta_1 H_1 \cos(\beta_1 d) = \beta_0 H_0 \cos(\beta_0 d + \Delta\phi), \tag{31}$$
$$H_1 \sin(\beta_1 d) = H_0 \sin(\beta_0 d + \Delta\phi), \tag{32}$$

where H_0 and H_1 are the magnetic field amplitudes in the z-direction in the cavity and in the sample, respectively. The phase difference $\Delta\phi$ can be determined from the change of the cavity length $\Delta L = L_0 - L_1$ that is needed in order to obtain the same resonance frequency as before the insertion of the sample:

$$\Delta\phi = \beta_0 \Delta L = \beta_0 (L_0 - L_1) \,. \tag{33}$$

Using (31) and (33), one obtains a transcendental equation for the unknown $x = \beta_1 d$

$$\frac{\tan x}{x} = \frac{\tan\left(\beta_0\left(d + \Delta L\right)\right)}{\beta_0 d} \,, \tag{34}$$

where the value for the expression on the right-hand side can be obtained by measurements. From (34), one can determine β_1 and with (30) one obtains ϵ_r.

4 Exercises

4.1 Preliminary Discussion

Please be prepared to discuss the following topics during the the preliminary discussion:

1. microwaves as a part of the electromagnetic spectrum,
2. the relationship between the wavelength and frequency in a dielectric material,
3. permittivity and permeability,
4. waveguides and waveguide modes,
5. waveguide resonators,
6. the measurement setup and procedure.

4.2 Measurements

It is recommended that the exercises 1 and 2 are completed before starting the measurements.

1. Determine the relation between $\frac{\Delta f}{f}$ and $\frac{\Delta L}{L_0}$ with help of (22) such that ϵ_r for TE modes is a function of $\frac{\Delta L}{L_0}$ according to (26). The following steps have to be carried out:
   (a) Insert $L_0 + \Delta L$ and L_0 into (22) and formulate an expression for $(f(L_0 + \Delta L) - f(L_0))/f(L_0)$. Now $\Delta f/f_0$ depends on $\Delta L/L_0$.
   (b) Simplify the expression and neglect the quadratic terms of ΔL.
   (c) Use the Taylor expansion of $1/(1+x)$ and remove the terms with quadratic or higher exponents. Which center point would make sense?
   (d) In order to obtain a linear relation between ΔL and Δf use the first order Taylor expansion of $\sqrt{1-x}$.
   (e) Substitute the resulting expression into (27) and use cylindrical coordinates to calculate the integrals. Calculate $|E_0|^2$ using the expression $E_r E_r^* + E_\phi E_\phi^*$ and the field components given by (25). Please note that a standing wave is considered. The z-dependence is determined by the vanishing field at $z = 0$ und $z = L_0$.

2. Compute the theoretical resonance frequencies of all modes in the frequency range 7–13 GHz for the cavity lengths of 30 mm, 40 mm and 60 mm.

3. Measure all resonances in the frequency range 7–13 GHz for the cavity lengths of 30 mm, 40 mm and 60 mm. Identify the associated modes and give a reason for non-appearing modes (see Appendix D).

4. Using the same cavity lengths, measure the frequency shift of the resonances of the modes TE_{011}, TE_{211}, and TE_{212} caused by the insertion of a sample into the cavity. Derive the permittivities of several samples by applying the perturbation theory and the exact solution.

5. Measure the lengths of the empty cavity L_0 so that the resonances of the modes TE_{011}, TE_{211}, and TE_{212} reside at a specific frequency in the range 9.4 GHz . . . 9.5 GHz. Fill the cavity with a sample and repeat the measurements to determine L_1 for each of the three modes. Compute ϵ_r of different samples using the perturbation theory and the exact solution.

6. Compute the quality factors

$$Q = \frac{f_r}{BW_{-3dB}}, \tag{35}$$

   where f_r is the resonance frequency and BW_{-3dB} the full width at half maximum (FWHM) of the resonance peaks, for the resonances obtained in the previous step. What is the meaning of the quality factor?

7. Identify and discuss sources of measurement uncertainty and perform an uncertainty analysis for each measurement. Which source is the most important one?

8. Compare the results obtained using different methods: frequency shift vs. change of cavity length, different modes, different cavity lengths, perturbation theory vs. exact solution and so on. Compare your results with reference values from the literature.

5 Instructions

5.1 Using the Vector Network Analyzer

Power on/off: Switch on the VNA units from bottom to top (the topmost unit ”85060C Electronic Calibration Unit” may remain switched off). Switch off the units in the reverse order.

Set frequency range: VNA - Stimulus - Start/Stop - Enter frequency with keypad - Select unit (G for GHz)

Set frequency resolution: VNA - Stimulus - Menu - Number of points (801 points recommended)

Averaging on/off: VNA - Response - Menu - Averaging on/off

Add marker: VNA - Menus - Marker - Select marker number (move the marker with the rotary knob)

Find peak: VNA - Menus - Marker - More - Market to target - Maximum

Restore default settings: VNA - Instrument state - Preset (ignore the displayed warning)

Save data to computer: Computer - Desktop - Icon ”S21” - Enter filename without extension (txt-extension will be added automatically)

Location of saved data: Computer - Desktop - Folder ”resonator_data”

File format (columnwise): f [GHz], Amplitude [dB], Phase [deg] (phase is not needed in this exercise)

5.2 Miscellaneous Notes

- The probe thickness d can be measured with a micrometer.
- There is an offset $L_{offset} = L - L_{scale} = 13.0 \pm 0.2$ mm between the actual cavity length and the value read from the micrometer scale of the resonator.
- When performing broadband measurements, be aware of the limited frequency resolution of the vector network analyzer.
- Show plots of the measurements in your report to illustrate how the data analysis was performed. Use of Matlab, Octave, or similar software for data processing is recommended.
- Prefer books and articles over websites when giving literature references.
- Handle the vector network analyzer and the cables with great care.
- Eating or drinking in the laboratory is not allowed.

References

[1] Unger, Hans-Georg: *Elektromagnetische Wellen I, II*, Friedr. Vieweg & Sohn, Braunschweig, 1967.

[2] Atwater, H. A.: *Introduction to microwave theory*, McGraw Hill, New York, 1962.

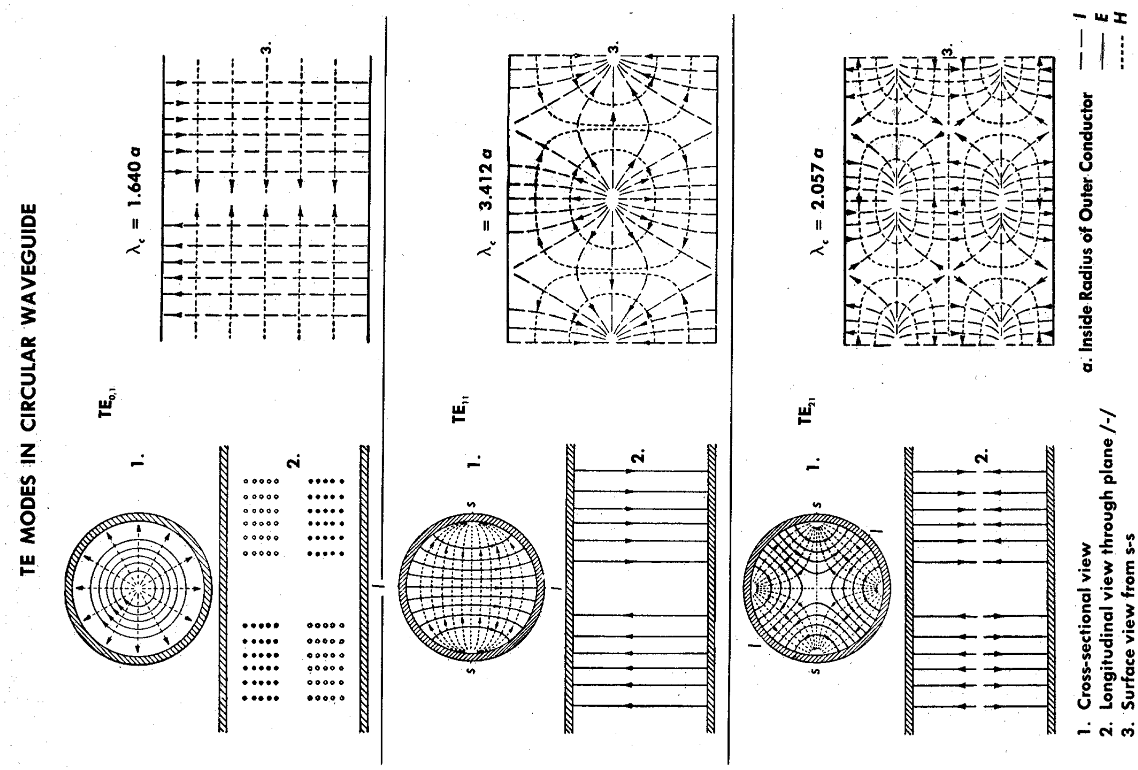
[3] Gardiol, Fred E.: *Introduction to microwaves*, Artech House, Dedham, 1984.

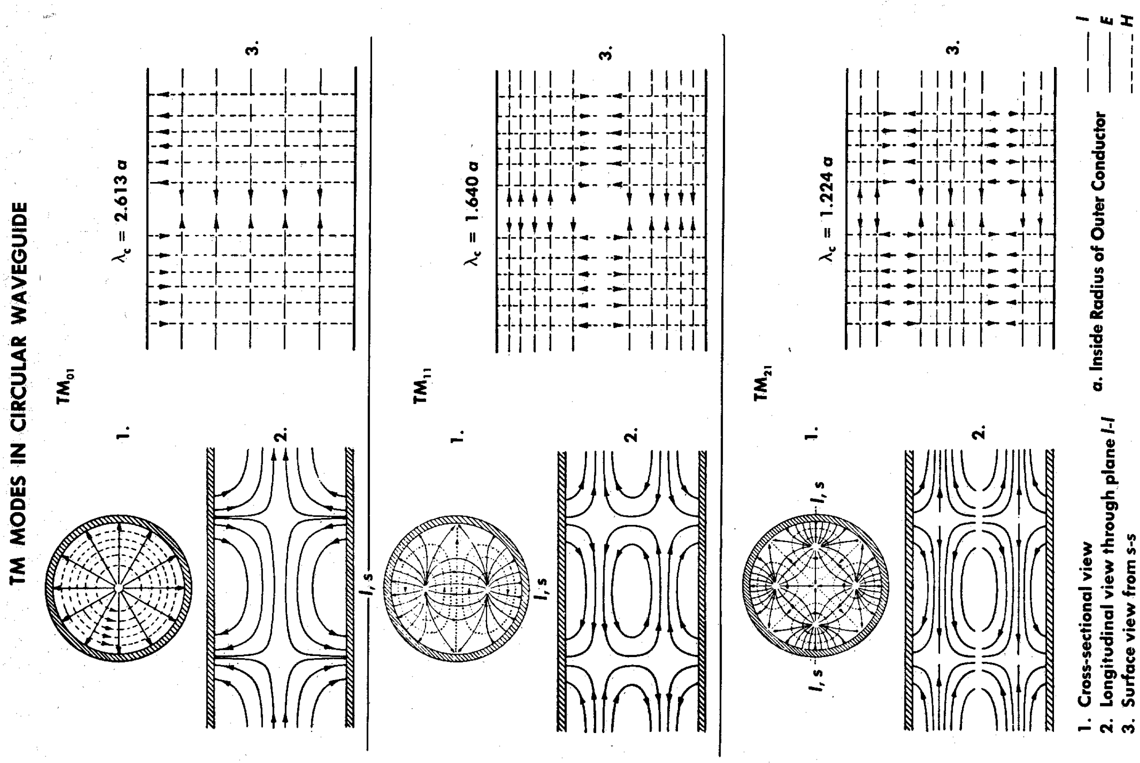
[4] Meinke/Grundbach: *Taschenbuch der Hochfrequenztechnik 1, 2, 3*, Springer, Berlin/ Heidelberg, 1986.

[5] Balanis, C. A.: *Advanced Engineering Electromagnetics, 2nd Edition*, Wiley, 2012.

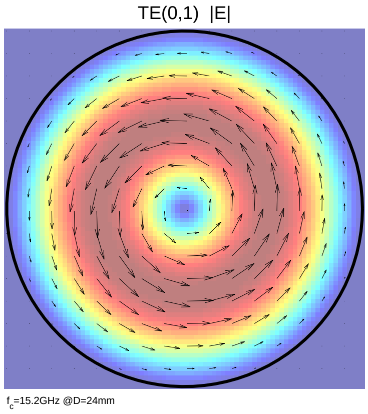
[6] Pozar, D. M.: *Microwave Engineering, 3rd Edition*, Wiley, 2005, sec. 6.7

A Field lines in a cylindrical waveguide

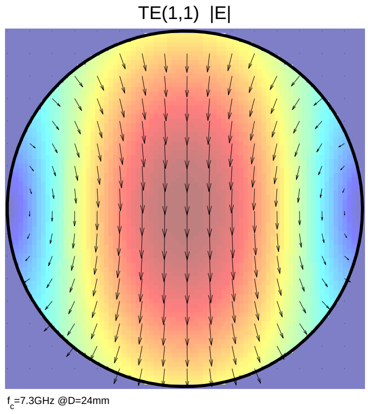




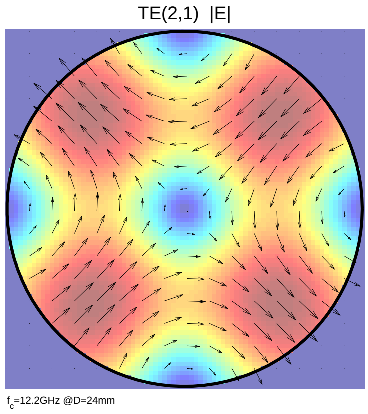
B Electric Field amplitudes in a cylindrical waveguide



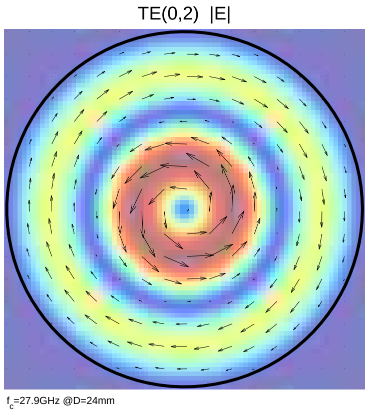
(a) TE01



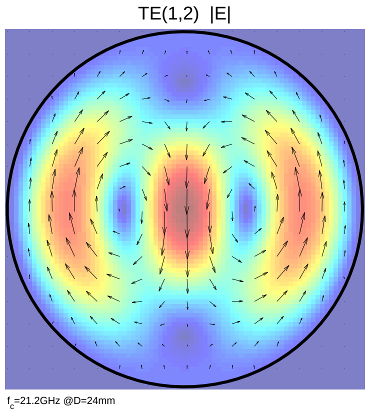
(b) TE11



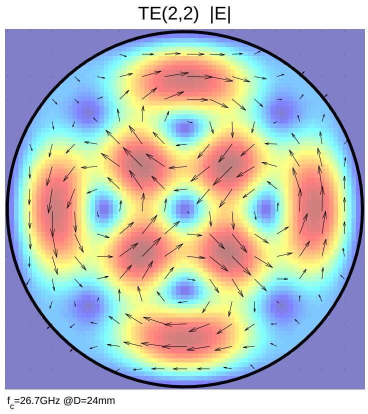
(c) TE21



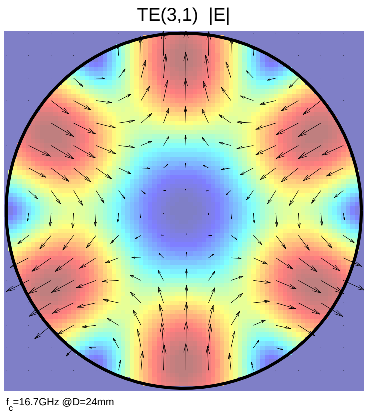
(d) TE02



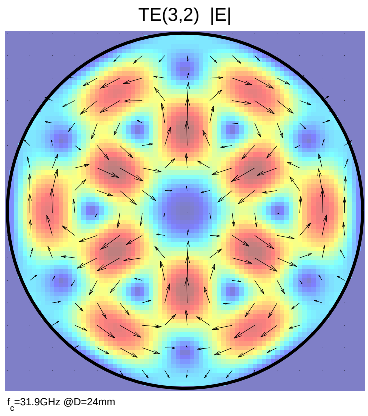
(e) TE12



(f) TE22

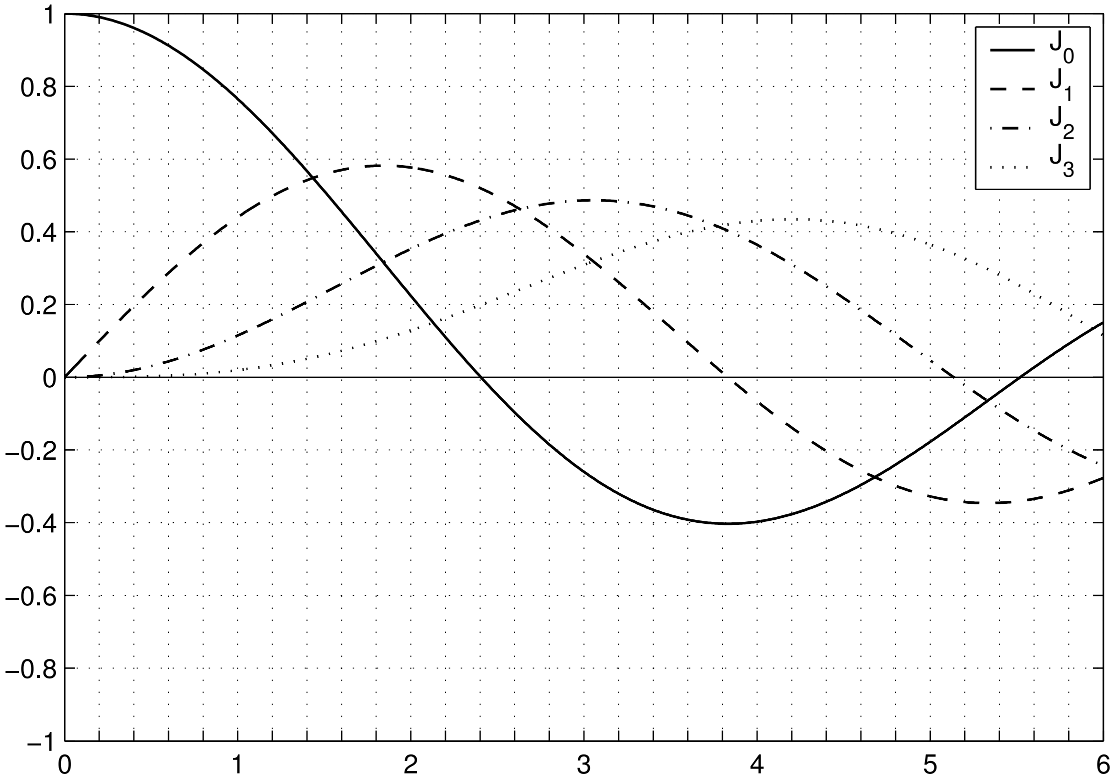


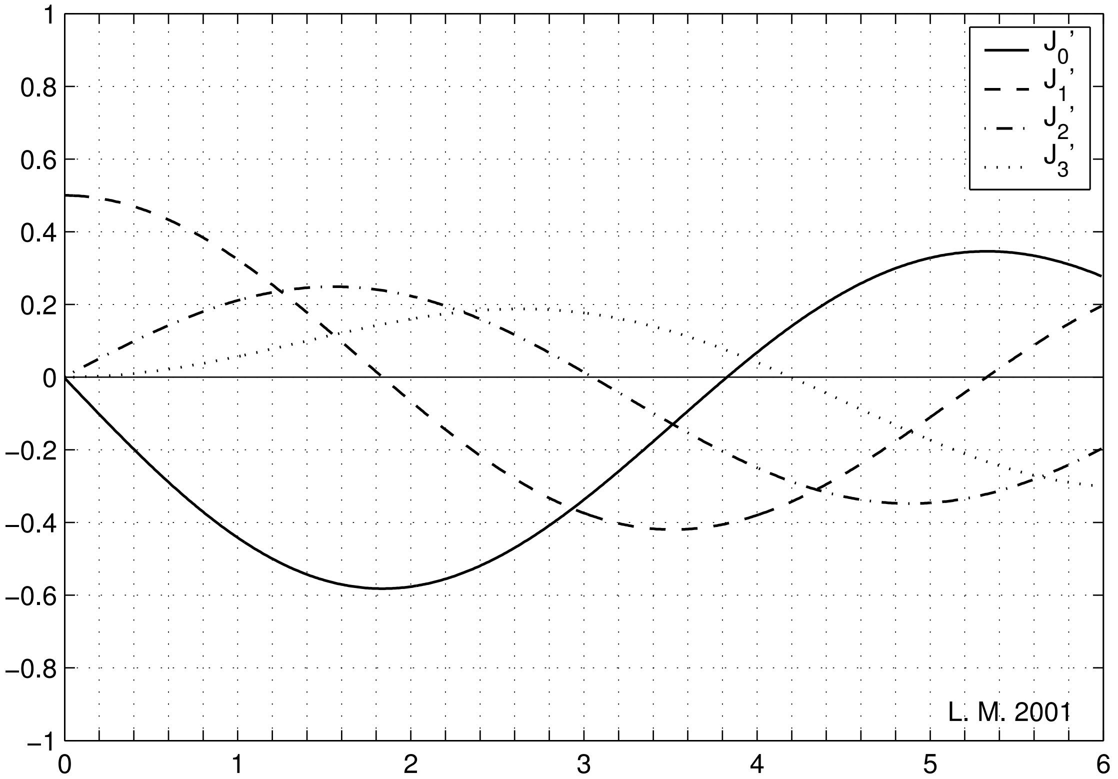
(g) TE31



(h) TE32

C Bessel Functions





D Resonant mode chart for a cylindrical cavity