Physics Institute
University of Bern

Laboratory exercises for
Modern Physics

Experiment instructions

Heat conduction

using Ångström's method

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0 Introduction

The aim of this experimental exercise is to measure the heat conductivity of copper using Ångström's method. This is carried out by periodically heating a copper rod with steam and cooling it with water. The resulting temperature profile along the rod is used for deriving the thermal conductivity of copper, which requires a sophisticated mathematical solution for an equation with partial derivatives using Fourier analysis.

In the first part of these instructions, the necessary background theory is described. This consists of a short description of the different types of simple partial differential equations, followed by the derivation of the heat equation, and a description of Angström's method.
The second part contains the experimental design, the required equipment, and information about the measuring procedure.

1 Theory

1.1 Overview of simple partial differential equations

A system with finite degrees of freedom (e.g. planetary system) is described by one or an endless number of functions of a variable (time, t). The equations of motion are ordinary differential equations, or systems of ordinary differential equations. To find the general solution for those equations, you also need the initial conditions.

A continuous system (e.g. homogeneous medium, string) is described by a function of several variables (e.g. x, y, z, t). The laws of nature appear in the form of partial differential equations, i.e. equations that combine the partial derivatives according to various of the variables x, y, z, t. In addition to the initial conditions, you also need boundary conditions if the spatial area is limited.

In contrast to the usual differential equations, a complete theory of partial differential equations does not (yet) exist. For physics and technology, certain linear partial differential equations - mostly of second order - are of particular importance.

A distinction is made between homogeneous linear partial differential equations, e.g.:

$$\frac{\partial u}{\partial t} = a^2 \Delta u \qquad \text{Heat equation} \tag{1}$$

where $u(x, y, z, t)$ is the temperature of a homogeneous medium,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{String equation} \tag{2}$$

where $u(x, y, z, t)$ is the deflection of a vibrating string fixed at the ends,

$$\Delta u = 0 \qquad \text{Laplace's equation (potential theory)} \tag{3}$$

where $u(x, y, z, t)$ is the potential in a vacuum between individual charges or stationary temperature distribution,

and inhomogeneous linear partial differential equations, e.g.:

$$\Delta u = f \qquad \text{Poisson's equation (potential theory)} \tag{4}$$

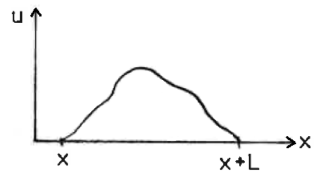
where f is a given function, independent of u.

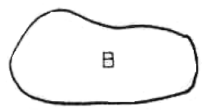
Similarly, there are homogeneous initial or boundary conditions, e.g.:

$$u(x, t) = u(x + L, t) \tag{5}$$

and inhomogeneous boundary conditions:

$$u(x, t) = f(x) \qquad (x \in B) \tag{6}$$





The set of solutions of a linear partial differential equation is obtained in the following way: First, one looks for the solutions of the homogeneous linear partial differential equation. These functions $u(x,t)$ must also fulfil the homogeneous ones under the given constraints. With the method of separating the variables:

$$u_n(x,t) = X(x) \cdot T(t) \tag{7}$$

i.e. the given homogeneous linear partial differential equation can be transformed into a system of ordinary differential equations for the functions $X(x)$ and $T(t)$. Then one attempts, by suitable superposition of the basic solutions, to obtain a function u that also satisfies the inhomogeneous constraints:

$$u = \sum_{n=0}^{\infty} a_n u_n \qquad (a_n \in \Re) \tag{8}$$

1.2 The heat equation

We here consider a rigid, isotropic and homogeneous heat conductor.

Terms:

c	:	specific heat capacity (for rigid bodies, $\mathrm{c_p} \approx \mathrm{c_v}$)
ρ	:	density
t	:	time
Q	:	amount of heat
$W(x, y, z, t)$	:	heat flux density vector
$U(x, y, z, t)$	:	temperature

The heat flux from an infinitesimal space element $\mathrm{d}\tau$ of mass $\mathrm{d}m$ is equal to the decrease in the heat content of $\mathrm{d}t$:

$$\nabla \cdot \overrightarrow{W} \mathrm{d}t = -\frac{\partial Q}{\partial t} \tag{9}$$

Every heat dissipation corresponds to a decrease in temperature:

$$dQ = c \, \mathrm{d}m \, \mathrm{d}u = c\rho \, \mathrm{d}\tau \, \mathrm{d}u \tag{10}$$

inserted in Eq. 9:

$$\nabla \cdot \overrightarrow{W} \mathrm{d}\tau = -c\rho \frac{\partial u}{\partial t} \mathrm{d}\tau \tag{11}$$

The law of heat conduction, also known as Fourier's law, states that the rate of heat transfer through a material is proportional to the negative gradient in the temperature:

$$\overrightarrow{W} = -k\nabla u \tag{12}$$

where k is the thermal conductivity.
From Eq. 11, one obtains:

$$-k\nabla u = -c\rho \frac{\partial u}{\partial t} \tag{13}$$

$$\boxed{\frac{\partial u}{\partial t} = K\Delta u} \tag{14}$$

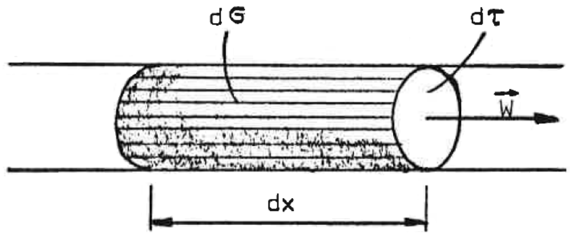
where:

$$K = \frac{k}{c\rho} \tag{15}$$

is the thermal diffusivity.

The heat conduction equation is a parabolic differential equation that also applies to other (compensation) processes such as diffusion, fluid friction, and electricity.

1.3 The heat conduction equation for a rod with radiation loss



Analogous to Eq. 9, the following applies:

$$S\,\mathrm{d}\sigma + \nabla \cdot \overrightarrow{W}\mathrm{d}\tau = -\frac{\partial Q}{\partial t} \tag{16}$$

the term $S\,\mathrm{d}\sigma$ taking the radiation loss on the surface of the rod into account.

S : radiation loss per surface unit $\mathrm{d}\sigma$ and time $\mathrm{d}t$

$\mathrm{d}\sigma$: surface element of the spatial element $\mathrm{d}\tau$

A linear relationship between S and u is assumed:

$$S = h \cdot u \tag{17}$$

where h is a measure of the radiation capacity of the surface.
The ambient temperature of the rod must now be defined, as the 'point zero' temperature, since S must disappear as soon as the rod element $\mathrm{d}\tau$ has the same temperature as the surroundings.

If one transforms Eq. 16 analogously to Eq. 9, one obtains:

$$h \cdot u\,\mathrm{d}\sigma - k\Delta u\,\mathrm{d}\tau = -c\rho\frac{\partial u}{\partial t}\mathrm{d}\tau \tag{18}$$

And with the substitutions:

$$\mathrm{d}\sigma = p\,\mathrm{d}x \qquad (\text{surface unit}: \; p\,\text{is the circumference of the rod}) \tag{19}$$

$$\mathrm{d}\tau = q\,\mathrm{d}x \qquad (\text{space unit}: \; q\,\text{is the crosssection of the rod}) \tag{20}$$

$$-\Delta u = \tfrac{\partial^2 u}{\partial x^2} \qquad (\text{as 1 dimentional}: u = u(x,t)) \tag{21}$$

one can derive the following:

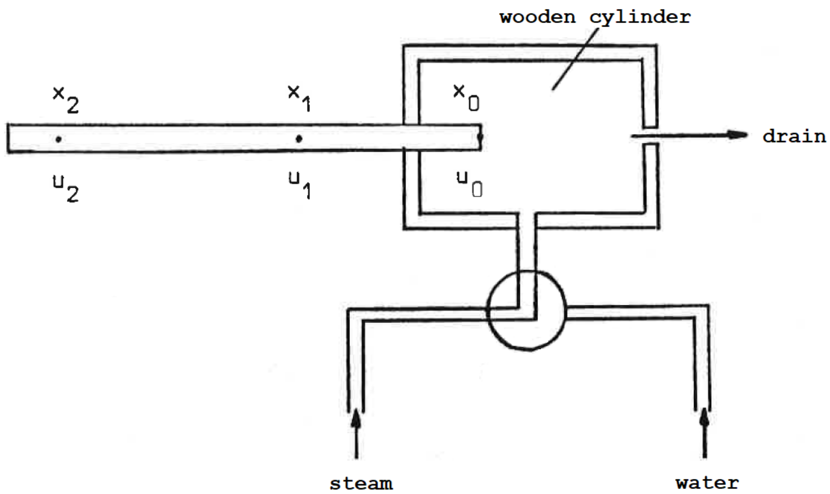
$$h \cdot u\,p\,\mathrm{d}x - k\tfrac{\partial^2 u}{\partial x^2}\,q\,\mathrm{d}x = -c\rho\tfrac{\partial u}{\partial t}q\,\mathrm{d}x \qquad \Leftrightarrow$$

$$\tfrac{\delta u}{\delta t} = \underbrace{\frac{k}{c\rho}}_{K}\;\tfrac{\delta^2 u}{\delta x^2} - \underbrace{\frac{h\,p}{c\rho q}}_{H}\;u \quad \Leftrightarrow$$

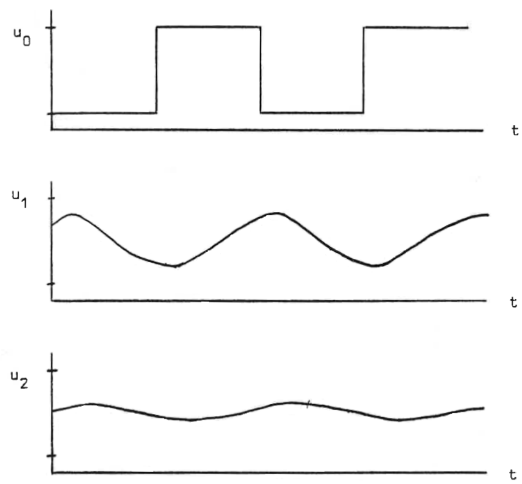
$$\boxed{\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} - H\,u} \tag{22}$$

1.4 Ångström's method

With this experimental method, one end of the rod is alternately heated and cooled with a period t until a periodic temperature profile is established in the rod.



The temperature profile at the three points x_0, x_1, x_2, are as follows:



For obvious reasons, it can be seen that with increasing distance from the hollow cylinder:
(a) the amplitude of the temperature fluctuations become smaller (reason?).
(b) the temperature changes are delayed (reason?).

The temperature (as a function of time) at two different points, x_1 and x_2, on the rod can be determined. First, the temperature, $u(x, t)$, has to be represented as a Fourier series. A Fourier analysis of the measured temperature gives the values for the amplitudes and phases of the individual Fourier terms. With the amplitude ratios and phase shifts of the respective Fourier terms from the two different measurement points, x_1 and x_2, the thermal conductivity k can be determined.

The periodic excitation function $f(t)$ at the end of the rod (at x_0) expands into a Fourier series:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{in\omega t} \tag{23}$$

where

$$\omega = \frac{2\pi}{T} \tag{24}$$

and c_n is complex.

For the temperature along the rod, the corresponding series is used:

$$u(x,t) = \sum_{n=-\infty}^{\infty} c_n u_n(x) \mathrm{e}^{in\omega t} \tag{25}$$

Each part of this series must individually satisfy the heat conduction equation Eq. 22. Because:

$$\frac{\partial^2 u_n}{\partial t^2} = in\omega u_n \tag{26}$$

becomes:

$$\frac{\partial^2 u_n}{\partial x^2} = \lambda_n^2 u_n \tag{27}$$

with

$$\lambda_n^2 = \frac{H + in\omega}{K} \tag{28}$$

The only physically meaningful solution to Eq. 27 is:

$$u_n(x) = \mathrm{e}^{-\lambda_n x} \tag{29}$$

As:

$$\lambda_n = \alpha_n + i\beta_n \tag{30}$$

then Eq. 25 becomes:

$$\boxed{u(x,t) = \sum_{n=-\infty}^{\infty} c_n \, \mathrm{e}^{-\alpha_n x} \, \mathrm{e}^{i(n\omega t - \beta_n x)}} \tag{31}$$

(Note: the same result would have been obtained if one had used the method of separating the variables with subsequent superposition).

With the coefficients α_n and β_n, the thermal conductivity can in principle be determined. From the two definitions Eq. 28 and 30 for λ_n one finds:

$$\lambda_n^2 = \alpha_n^2 + 2i\alpha_n\beta_n - \beta_n^2 = \frac{H}{K} + i\frac{n\omega}{K} \tag{32}$$

It follows from:

$$\beta_0 = 0, \, \alpha_0 = \sqrt{\frac{H}{K}} \tag{33}$$

$$\alpha_n^2 - \beta_n^2 = \frac{H}{K} \tag{34}$$

$$\alpha_n\beta_n = \frac{n\omega}{2K} = \frac{n\pi}{KT} \tag{35}$$

using Eq. 24. With Eq. 15 one gets:

$$\boxed{k = \frac{n\pi cp}{\alpha_n\beta_n T}} \qquad (k > 0) \tag{36}$$

Eq. 31 is now transformed in order to determine values for α_n, β_n from the Fourier analysis.

From a Eq. 34 and 35, one finds that:

$$\alpha_n = \sqrt{\sqrt{\frac{n^2\pi^2}{K^2T^2} + \frac{H^2}{4K^2}} + \frac{H}{2K}} \tag{37}$$

$$\beta_n = \sqrt{\sqrt{\frac{n^2\pi^2}{K^2T^2} + \frac{H^2}{4K^2}} - \frac{H}{2K}} \tag{38}$$

because of α_n, $\beta_n \geq 0$ it follows from Eq. 36 that $n \geq 0$.With $c_n = a_n e^{i\gamma_n}$, Eq. 31 then becomes:

$$u(x,t) = \sum_{n=-\infty}^{\infty} a_n \mathrm{e}^{-\alpha_n x} \mathrm{e}^{i(n\omega t - \beta_n x + \gamma_n)} \tag{39}$$

Using Euler's equation, $\mathrm{e}^{i\theta} = \cos\theta + i\,\sin\theta$, Eq. 39 becomes:

$$u(x,t) = a_0 \mathrm{e}^{-\sqrt{\frac{H}{K}}x} + \sum_{n=-\infty}^{\infty} a_n \mathrm{e}^{-\alpha_n x} \left[\cos\left(\frac{2\pi n}{T}t - \beta_n x + \gamma_n\right) + i\sin\left(\frac{2\pi n}{T}t - \beta_n x + \gamma_n\right)\right] \tag{40}$$

and for only the real part:

$$\boxed{u(x,t) = a_0 \mathrm{e}^{-\sqrt{\frac{H}{K}}x} \quad + \textstyle\sum_{n=-\infty}^{\infty} \quad \underbrace{a_n \mathrm{e}^{-\alpha_n x}}_{\text{amplitude}} \quad \cos\left(\frac{2\pi n}{T}t \quad \underbrace{-\beta_n x + \gamma_n}_{\text{phase}}\right)} \tag{41}$$

As Eq. 41 shows, every Fourier term has an amplitude, $a_n \mathrm{e}^{-\alpha_n x}$, and a phase, $-\beta_n x + \gamma_n$, which contain α_n, β_n.

The temperature curve over time is measured at two points, $x_1 = x'$ and $x_2 = x''$ (with l distance between them), of the rod, where:

$$x'' = x' + l \tag{42}$$

These two data sets are used in the Fourier analyses as temperature values, $u(x', t)$ or $u(x", t)$, and time values (which exactly correspond to one period) The values from the Fourier analyses gives the amplitudes and phases from the Fourier terms:

$$\text{Amplitude} \quad : \quad b'_n = a_n \mathrm{e}^{-\alpha_n x'} \qquad b''_n = a_n \mathrm{e}^{-\alpha_n x''} \tag{43}$$

$$\text{Phase} \quad : \quad \phi'_n = -\beta_n x' + \gamma_n \qquad \phi''_n = -\beta_n x'' + \gamma_n \tag{44}$$

From Eq. 43 one can find the ratio of amplitudes using Eq. 42:

$$\frac{b_n}{b'_n} = \mathrm{e}^{-\alpha_n l} \mathrm{e}^{-A_n} \tag{45}$$

where:$A_n = ln\left(\frac{b_n}{b'_n}\right) = \alpha_n l$. From Eq. 44 with 41 the phase shift $\Delta\phi_n$ and thereby β_n results in:

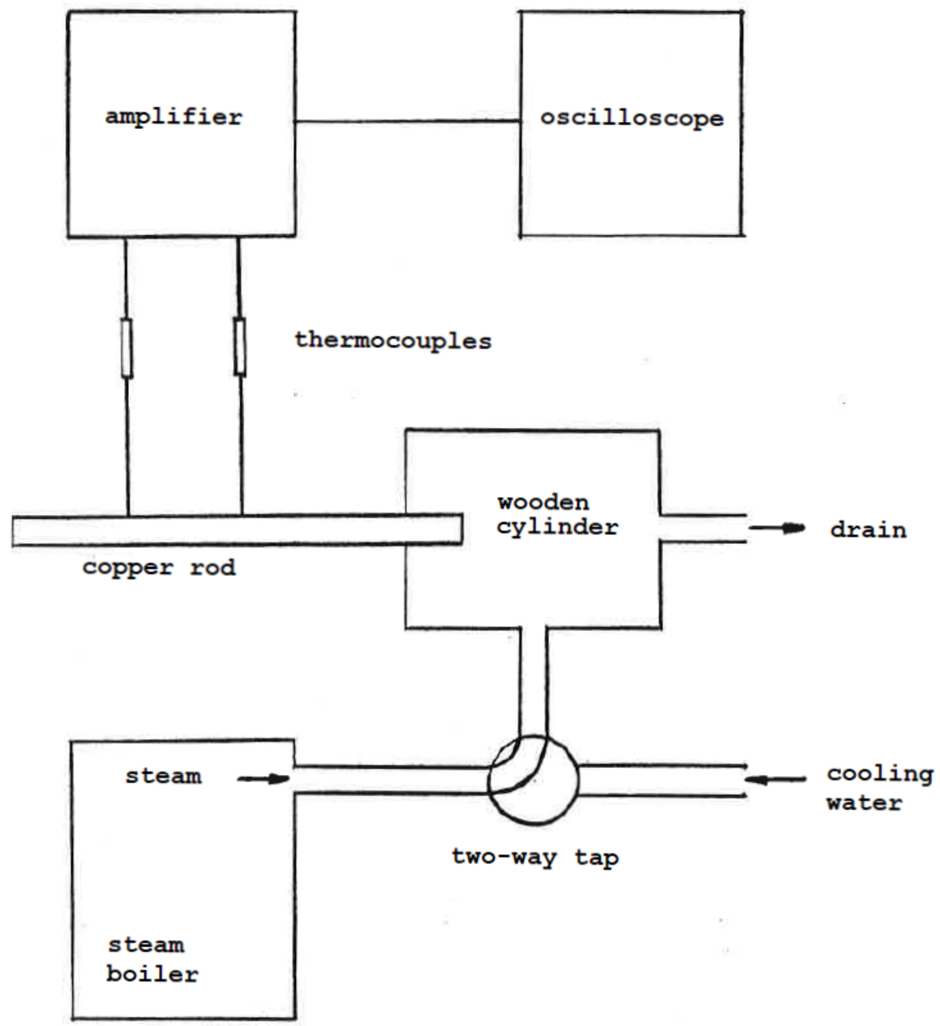
$$\Delta\phi_n = \phi'_n - \phi''_n = \beta_n l \tag{46}$$

Solved for respective α_n or β, and inserted in 36, one gets the **thermal conductivity** with the amplitude ratios 45 and phase shifts 46 from the Fourier analyses:

$$\boxed{k = \frac{n\pi c\rho l^2}{A_n \Delta\phi T}} \tag{47}$$

2 The experiment: Ångström's method

2.1 Experimental setup



2.2 Calibration of the thermocouples

2.2.1 Measurement equipment

A. Thermocouples Iron-constan thermocouples are used to measure the temperature of the rod. The Seebeck effect applies for the thermocouple; if two wires made of different metals are soldered together at both ends, a thermoelectric voltage U_{therm}, can be measured on an interposed voltmeter, so that the two soldering points are brought to different temperatures (u and u_0). For small temperature differences the following approximation applies:

$$U_{therm} = a(u - u_0) + b(u - u_0)^2 \tag{48}$$

The quantity

$$\eta = \frac{\partial U_{term}}{\partial u} = a + 2b\,(u - u_0) \tag{49}$$

is called the sensitivity or thermal force of the thermocouple. For not too large temperature intervals, then $b \approx 0$, i.e.

$$\eta \approx a \tag{50}$$

$$\boxed{U_{therm} \approx a\,(u - u_0)} \tag{51}$$

The thermoelectric voltage is therefore proportional to the temperature difference. The following applies to an iron-constan thermocouple:

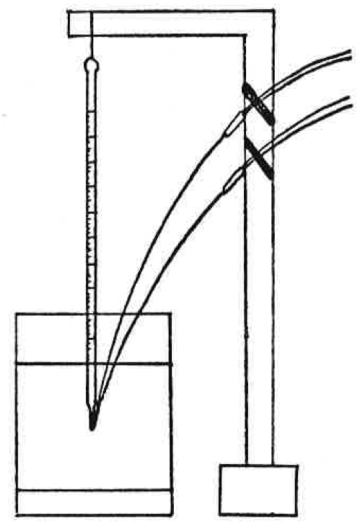
$$a = 53 \cdot 10^{-6}\,{}^{V}/_{^\circ\mathrm{C}} \qquad (a < 300\,^\circ\mathrm{C}) \tag{52}$$

B. Amplifier The measured thermal voltage is small and must therefore be amplified in order to read out the voltage outputs. The amplifier has two inputs for the thermocouples, and is connected to the oscilloscope.

C. Oscilloscope The oscilloscope is used to read out the voltage signal received from the thermocouples as it varies over time. The screen of the oscilloscope can be adjusted both vertically and horizontally, so that one can zoom in and out on the voltage signal. The trigger is useful to focus the display on the expected signal. The cursors can thereafter be used to read the signal values (as amplitude and phase).

2.2.2 Measurement procedure

In order to find the linear relationship between u and U_{therm} described in Eq. 51, one compares the temperature measured with a thermal mercury thermometer with the measured voltage excursion from a thermocouple.



As shown in the drawing, one holds the mercury thermometer and the soldering point of the loaded thermocouples as close together as possible in a vessel with water. This is now brought to the boil, making sure that the soldering point in the air (by the handle) is isolated from the water vapour. With a sufficient number of temperature readings (e.i. every 5 degrees), the associated voltage excursion of the thermocouples is measured by switching from one thermocouple to the other, both during heating and cooling.

Now the temperature is plotted against the voltage (in any recorder units A), and a linear calibration curve can be drawn through the recorded points:

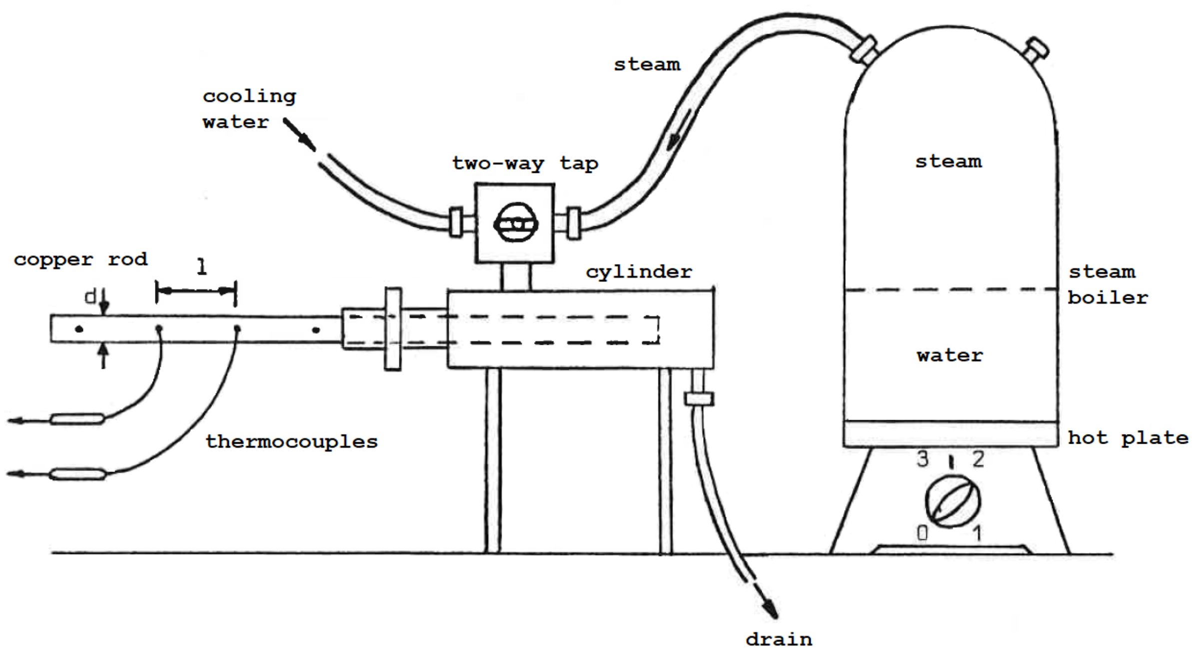
$$u(A) = b + c \cdot A \tag{53}$$

in ^oC.

With this method, one does not need to know the exact amplification, linearity, etc. of the voltage amplifier and recorder. The reference temperature u_0 is measured, also by the handles of the thermocouples in the air, and is contained in b, as can be seen from Eq. 51 and 53.

2.3 Measurement of the thermal conductivity of copper using Ångström's method

2.3.1 Measurement setup



One end of the copper rod is in a hollow cylinder and is screwed tight with a gasket. Through the hollow cylinder, one can conduct steam from a steam boiler or cooling water by means of the shut-off valve. The copper rod (diameter) has four holes for the thermocouples.

The thermocouples can be placed in different configurations in the holes (in the figure they are in the two middle holes).

2.3.2 Measurement procedure

The end of the copper rod is heated and cooled periodically (exactly $\frac{1}{2}$ period steam, $\frac{1}{2}$ period cooling water). Reasonable period lengths are about T = 240 s (2 min steam, 2 min water) or T = 360 s (3 min steam, 3 min water). When the signal reaches (temperature) equilibrium (after approx. 6 periods or more until the curve becomes regular) the temperature profile is recorded at both measuring points simultaneously.

3 Exercises

As the present experiment "heat conduction" focuses less on the physical measurement than on the mathematical evaluation, it is particularly important that the theory is well understood. Therefore, the following exercise must be solved before the start of measurements:

- Solve the heat conduction equation 22 with the method of separating the variables and show that equation 31 is obtained by suitable substitutions.

The following measurements must be carried out:

- A calibration curve is to be recorded from the two provided thermocouples, and the linearity $u(A) = b + cA$ is to be shown in the final report. The coefficients b and c are to be determined (linear regression).
- The temperature profile should be recorded at two points using the two thermocouples in the holes in the copper rod during one full period (exactly $\frac{1}{2}$ period steam, $\frac{1}{2}$ period cooling water). Two measurements are made:
  a) with the thick rod (T = 360 sec)
  b) with the thin rod (T = 240 sec)

For the evaluation:

- Make a Fourier analysis of the periodic functions obtained and determine the coefficient of thermal conductivity k. Think about how many harmonics you should take into account and try to estimate the error. The evaluation can either be done by hand (quite laborious) or carried out with a program.