IAP: Laborkurs moderne Physik

Fourier Optics

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Contents

1. Laser Safety

The laser we use in this experiment is a Helium-Neon gas laser (Uniphase, U.S.) with a wavelength of 633nm, and an output power of 0.95 mW, which is classified as a CLASS 2 LASER PRODUCT.

A Class 2 laser is safe because the blink reflex will limit the exposure to no more than 0.25 seconds. It only applies to visible-light lasers (400–700 nm). Class-2 lasers are limited to 1 mW continuous wave, or more if the emission time is less than 0.25 seconds or if the light is not spatially coherent. Intentional suppression of the blink reflex could lead to eye injury.

2. Introduction

Ray optics is a convenient tool to determine imaging characteristics such as the location and magnification of the image. But ray optics cannot provide the wave properties of light and associated processes like diffraction which are essential to a complete description of the imaging system, because these processes determine the image contrast, the resolution of optical devices, and the effect of spatial filters. One possible wave-optical treatment considers the Fourier spectrum (space of spatial frequencies) of the object and the transmission of the spectral components through the optical system. This is referred to Fourier optics.

In Fourier optics, Fourier transforms are used to study optical phenomena. The physical concept is based on the wavelike character of light. By regarding every light-wave as a superposition of plane waves it turns out that optical image formation can easily be treated within the mathematical framework of Fourier methods. This is often used in image processing, spatial filtering and Holography. In this laboratory we will use a He-Ne-laser and a set of lenses to visualize and manipulate Fourier transformed images.

In this lab you will learn the principles of spatial filtering to visualize and manipulate Fourier transformed images with a He-Ne-laser and a set of lenses.

3. Theory

3.1 Mathematical concepts

3.1.1 Fourier series

The basic Fourier theorem states: A periodic function $f(t)$, with angular frequency ω, can be considered as the sum of harmonic functions whose frequencies are multiples of ω, where ω is called the fundamental frequency. This can be mathematically expressed as:

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t) \tag{1}$$

or,

$$f(t) = \sum_{n=-\infty}^{+\infty} c_n \, e^{in\omega t} \tag{2}$$

The amplitude of each term of the series can be calculated using the following relations:

$$a_0 = \frac{1}{T}\int_0^T f(t)\mathrm{d}t \tag{3}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t \,\mathrm{d}t \tag{4}$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t \,\mathrm{d}t \tag{5}$$

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-in\omega t}\mathrm{d}t \tag{6}$$

where $T = \frac{2\pi}{\omega}$ is the period of $f(t)$.

While in our case we are not dealing with time dependent but rather with position dependent functions, we will use the spatial position parameter x instead of t, the spatial frequency k instead of the angular frequency ω and the wavelength λ instead of the period T.

3.1.2 Fourier transform

A more general definition considers a non-periodic function $f(x)$. In this case, Fourier series need to be replaced by Fourier integrals. This can be understood by thinking of a non-periodic function as a function with an infinite period. Therefore, to express such a function as a Fourier decomposition, it is necessary to sum over a continuous spectrum of frequencies.

$$f(x) = \int_{-\infty}^{\infty} F(k) e^{2\pi ikx} \, \mathrm{d}k \tag{7}$$

with:

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi ikx} \, \mathrm{d}x \tag{8}$$

3.1.3 Convolution

The convolution of two functions f and g is defined as the integral of the product of the two functions after one is reversed and shifted. The result of this so called integral transform is a third function giving the area of overlap between the two functions f and g as a function of the amount that one of them is translated:

$$h(X) = f(x) \otimes g(x) = \int_{-\infty}^{\infty} f(x) \, g(X - x) \mathrm{d}(x) \tag{9}$$

Taking the Fourier transform of these functions it becomes clear that the convolution corresponds to a simple multiplication in Fourier space, i.e.:

$$H(k) = F(k) \cdot G(k) \tag{10}$$

This knowledge will be very important when doing the experiments.

3.2 Physical concepts

3.2.1 Diffraction

Diffraction can be defined as 'any deviation of a light ray from rectilinear propagation, which is not caused by reflection nor refraction'. It was already known for centuries that light rays, passing through a small aperture in an opaque screen do not form a sharp shadow on a distant screen. That smooth transition from light to shadow could only be explained by assuming that light has a wavelike character. Diffraction theory has been further developed by Huygens, Fresnel, Kirchhoff and Sommerfield.

An important starting point for understanding diffraction phenomena and gaining insight in the theory of Fourier optics is the Huygens-Fresnel principle. According to this principle, each point of a wavefront (regardless whether its a light- or mechanical wave) acts as a punctual (light)source. The resulting wave is the sum of each of those 'wavelets'. The Huygens-Fresnel principle is illustrated for small and large apertures in Figure 1.

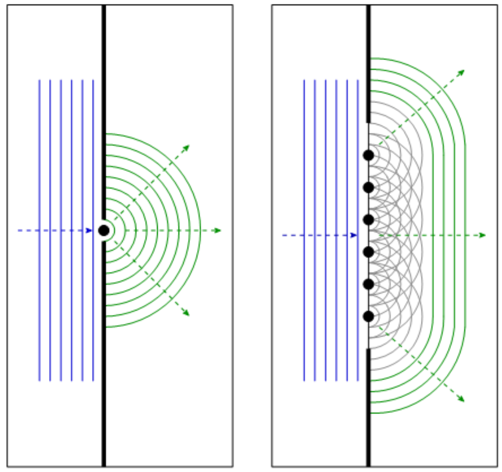

Figure 1: Illustration of diffraction based on the Huygens-Fresnel principle for small (left) and large(right) apertures[2].

We can now turn to a more quantitative formulation of the Huygens-Fresnel principle. Each of the punctual sources mentioned in the principle will emit spherical waves that decrease in amplitude as the distance increases. These waves can be described by the following formula:

$$\Psi_{spherical} = \frac{A}{r} e^{ikr} \tag{11}$$

with A the initial amplitude, r the distance from the source and $k = \frac{2\pi}{\lambda}$ the wavenumber.

It is useful to note that in these laboratory notes, we will limit ourselves to the approximation of a scalar theory. We will only consider one single component of the electric or magnetic field vector. This also means that we neglect the (possible) coupling between electric and magnetic fields. By comparing this approximation with exact theories and experiments, it turns out that this scalar diffraction theory is good whenever the diffracting aperture is large compared with the wavelength of the light, and the diffracting field is calculated at a large distance from the aperture.

To calculate the diffraction pattern produced by an arbitrary aperture $A(x;\ y)$ at a distance z away from this aperture, we have to take the sum of the waves produced by punctual sources at every point of the aperture. This results in the following integral:

$$A'(x', y') = \frac{1}{i\lambda} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(x, y) \frac{z}{r} \frac{e^{ikr}}{r} \mathrm{d}x\, \mathrm{d}y \tag{12}$$

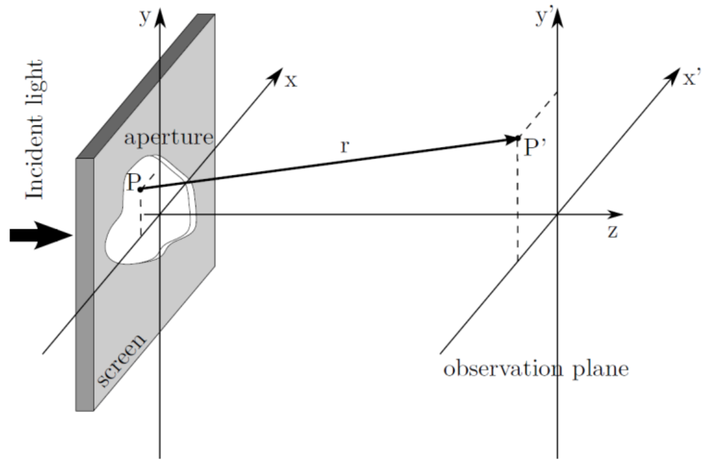

Figure 2: Diffraction of the light incident on an aperture.

where, $r = \sqrt{(x-x')^2 + (y-y')^2 + z^2}$ is the distance between a given point $(x;\ y)$ of the aperture and a point $(x_0;\ y_0)$ of the observation plane. We will not focus on the derivation of this formula in which the factor $\frac{1}{i\lambda}$ comes up. The factor of $\frac{z}{r}$ corresponds to the cosine of the angle between the vectors r and z. This diffraction integral is a special case of the Fresnel-Kirchho diffraction formula REF.

3.2.2 Fresnel and Fraunhofer approximation

Now we will simplify the diffraction integral (12) further by making some additional assumptions. Suppose that the axial distance *z* is much larger than the transverse dimensions. Then

$$\cos(z,r) = \frac{z}{r} \cong 1 \tag{13}$$

The error is smaller than 5%, when the angle is smaller than 18°. Also the remaining r in the denominator of (12) may be replaced by z. In the exponential we have to be a little bit more careful since in a phase term, small differences have a bigger effect. Instead of replacing r by z we rather develop it in a binomial expansion, retaining only the first two terms:

$$\begin{aligned} r &= \sqrt{(x-x')^2 + (y-y')^2 + z^2} \\ &\cong z\left[1 + \frac{1}{2}\left(\frac{x-x'}{2}\right)^2 + \frac{1}{2}\left(\frac{y-y'}{2}\right)^2\right] \\ &\cong z + \frac{x^2+y^2}{2z} + \frac{xx'+yy'}{2z} + \frac{x'^2+y'^2}{2z} \end{aligned} \tag{14}$$

Implementing this in equation (12) results in the so called Fresnel diffraction integral:

$$A'(x',y') = \frac{e^{ikz}}{i\lambda z} e^{\frac{ik}{2z}(x'^2+y'^2)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(x,y) e^{\frac{ik}{2z}(x^2+y^2)} e^{-\frac{i2\pi}{\lambda z}(xx'+yy')} \mathrm{d}x\,\mathrm{d}y \quad (15)$$

The integral can be interpreted as a 2-dimensional Fourier transform of $A(x,y)e^{\frac{ik}{2z}(x^2+y^2)}$ with spatial frequencies $f_x = -\frac{x'}{\lambda z}$ and $f_y = -\frac{y'}{\lambda z}$.

As this result is valid close to the aperture, it is called the near-field approximation. This approximation is however not valid too close to the aperture, and it is not easy to calculate exactly the limits of validity. A sufficient condition is that the higher order term in the expansion be small, but this is not a necessary condition. Indeed, it suffices that they do not change the value of the integral too much after integration, and this also depends on the aperture function $A(x;\ y)$. The general conclusion of deeper analyses is that the accuracy of the Fresnel approximation is extremely good to distances that are very close to the aperture.

Whereas in the Fresnel regime, we still take into account that the wavefronts of the point sources are curved (by using the binomial expansion up to 2 terms in the exponential). If we go even further away from the screen, that is not necessary anymore. Indeed, for $z \to \infty$ we can assume that $\frac{x^2+y^2}{z} \to 0$. This results in the Fraunhofer diffraction integral:

$$\begin{aligned} A'(x',y') &= \frac{e^{ikz}}{i\lambda z} e^{\frac{ik}{2z}(x'^2+y'^2)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(x,y) e^{-\frac{i2\pi}{\lambda z}(xx'+yy')} \mathrm{d}x\,\mathrm{d}y \\ &= \frac{e^{ikz}}{i\lambda z} e^{i\pi\lambda({f_x}^2+{f_y}^2)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(x,y) e^{-\frac{i2\pi}{\lambda z}(xx'+yy')} \mathrm{d}x\,\mathrm{d}y \end{aligned} \quad (16)$$

Since we will only observe the intensity of the signal, the phase terms in front of the integrals can be considered to be constant and do not have to bother us at all. This approximation is valid as long as

$$z >> \frac{k(x^2+y^2)_{max}}{2} \quad (17)$$

And it clearly shows that the field in the image plane can be interpreted as the Fourier transform of the field in the object plane (in this case the aperture).

3.2.3 Lenses in Fourier systems

- Intuitive understanding from geometrical optics

Fourier optics is a very mathematical and abstract field in optics. To facilitate understanding, it is therefore useful to start with some intuitive notes (which are by no means meant to be a correct and complete physical model), based on well-known geometrical optics principles.

Lenses transform angles to position and vice versa. A point source in the front focal plane of the lens results in parallel rays behind the lens. A ray with a large angle before the lens becomes a parallel ray far away from the center in the back focal plane. A ray with a small angle before the lens ends up close to the center in the back focal plane.

On the other hand, when a parallel beam is incident on a lens, the rays will be focused in the back focal plane. The ray that goes through the center of the front focal plane will have an angle of zero degrees after the lens. The further away the rays are from the center (in the front focal plane), the larger the angle in the back focal plane.

In both cases it appears that, when comparing the situation in the front and the back focal plane of the lens, angles have been transformed to position and vice versa. This is illustrated in figure 3. Also, the lateral position of the focus in figure 3 d depends on the angle of the incident plane waves. Again this can be turned around. The lateral position of the point source in figure 3 c determines the angle of the parallel beam behind the lens.

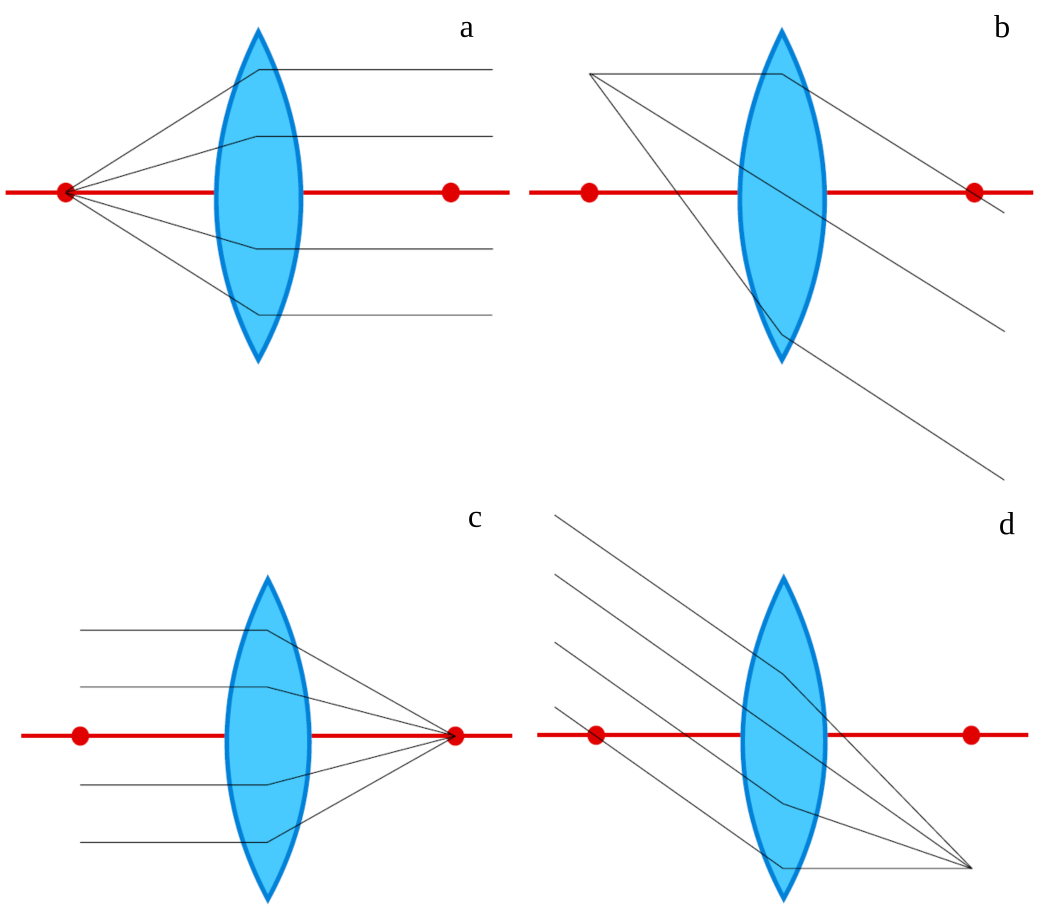

Figure 3: Lenses in geometrical optics: a. and b. The different angles in the front focal plane result in different positions in the back focal plane. c. and d. The different positions in the front focal plane result in different angle at the back focal plane.

In Fourier optics every optical wave is considered as a sum of plane waves with different angles, this is also called the angular spectrum of plane waves. A plane wave translates to a set of parallel rays in geometrical optics. Considering the previous we can assume that this spectrum of plane waves will be depicted as different lateral positions in the back focal plane of the lens.

Infinity is brought to the focal plane. Let us consider two parallel rays. We could say these rays only cross at infinity. We can make those rays cross at a finite distance by adding a lens. They will now cross in the back focal plane of the lens. The lens has brought a point at infinity to its focal plane. From the previous section we know that the Fraunhofer regime, in which the obtained field is the Fourier transform of the original field distribution, is only obtained at infinity. However a lens seams to bring points at infinity to its focal plane, so that we expect to find the Fourier transformed field of whatever is in the front focal plane, in the back focal plane of the lens.

- Phase transformation by a lens

In this section we calculate the phase transform a plane monochromatic wave undergoes when it passes through a lens. We will assume a thin lens, which means the lateral position of the light ray is the same in the back tangent plane ($\boldsymbol{\sigma'}$) as in the front one ($\boldsymbol{\sigma}$). The only thing that happens is that inside the lens, which is made of a material (glass) with refractive index $\boldsymbol{n}$, the light travels slower than in air. This retardation of the light is proportional to the local thickness $\Delta(\boldsymbol{x},\boldsymbol{y})$ of the lens. If we call $\Delta_{\boldsymbol{max}}$ the maximum thickness of the lens, we can write the phase retardation between the two tangent planes as:

$$\phi(x,y) = kn\Delta(x,y) + k(\Delta_{max}\text{-}\Delta(x,y)) \tag{18}$$

In which the first term represents the propagation through the lens, and the second term the propagation through the air.

For calculating $\Delta(x;\ y)$ we will split up the lens in two parts as shown in figure 4. The radius of curvature R is defined such that it is positive for a concave surface, and negative for a convex one.

$$\Delta(x,y) = \Delta_1(x,y) + \Delta_2(x,y) \tag{19}$$

and,

$$\Delta_{max} = \Delta_{max,1} + \Delta_{max,2} \tag{20}$$

we get:

$$\begin{aligned}\Delta_1(x,y) &= \Delta_{max,1} - \left(R_1 - \sqrt{R_1^2 - x^2 - y^2}\right) \\ &= \Delta_{max,1} - R_1\left(1 - \sqrt{1 - \frac{x^2+y^2}{R_1^2}}\right)\end{aligned} \tag{21}$$

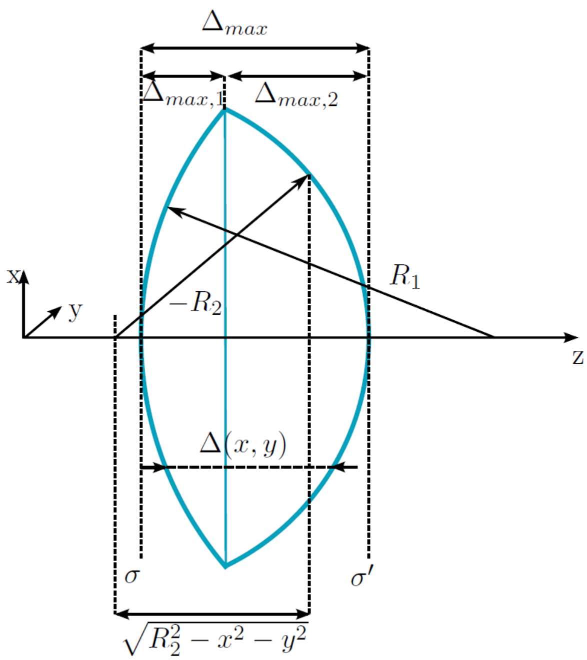

Figure 4: Parameters for the calculation of the phase delay of a thing lens.

and

$$\Delta_2(x,y) = \Delta_{max,2} - \left(-R_2 - \sqrt{R_2^2 - x^2 - y^2}\right)$$
$$= \Delta_{max,2} + R_2\left(1 - \sqrt{1 - \frac{x^2 + y^2}{R_2^2}}\right) \tag{22}$$

These expressions can be simplified in the paraxial approximation:

$$\sqrt{1 - \frac{x^2 + y^2}{R^2}} \cong 1 - \frac{x^2 + y^2}{R^2} \tag{23}$$

hence

$$\Delta(x,y) = \Delta_{max} - \frac{x^2 + y^2}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{24}$$

If we call A_l the field at the front tangent plane σ, and $A_{l'}$ the field in the back tangent plane σ',we can write

$$A_{l'} = e^{ikn\Delta_{max}} e^{-ik(n-1)\frac{x^2+y^2}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)} A_l \tag{25}$$

Now we can define a single parameter f, which we call focal distance to combine all physical parameters n, R_1, R_2 from the lens:

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{26}$$

So that the final transformation by the lens in the paraxial approximation looks like

$$A_{l'} = e^{ikn\Delta_{max}} e^{-i\frac{k}{2f}(x^2+y^2)} A_l \tag{27}$$

- Fourier transforming properties of lenses

The calculations belonging to this section will be up to you as part of the theoretical exercises. You will consider a certain field distribution at a distance d in front of a thin lens met focal length f, and calculate the resulting field in the back focal plane of this lens. You may assume that distances d and f are large enough so that the Fresnel diffraction regime is valid. Be sure to check the case $d = f$.

3.3 Theoretical exercises

Before starting the lab work, it is important to have read and understood the theory. These exercises check your understanding of the mathematical and physical concepts behind Fourier optics. You are also challenged to think about how to apply all this in a lab situation.

Mathematical concepts of Fourier transformation

- Produce a graphical representation of the Fourier transform of a rectangular pulse. Comment on the relation between the pulse width and the distance between the maxima of its Fourier transform.
- Given are two rectangular pulses $f(x)$ and $g(x)$, with an amplitude 1 and a width of 4π. Plot those functions and their convolution in one plot. In another plot, display the Fourier transforms of all 3 functions. Comment.

Physical concepts of Fourier optics

- With a simple setup containing an aperture, a laser (with beam expander) and a screen, the Fraunhofer or far-field image of the aperture is displayed on the screen. How can we, without changing the positions of the light source, aperture and screen, change to the near-

field or Fresnel image of this aperture on the screen? What do you expect when your proposed change is pushed to the limit?
- Calculate how an image at a distance gets transformed by a lens (paragraph 2.2.3.3). Explain each step.

Theoretical knowledge application

- Look at the setup depicted in figure 5. Explain what happens in this setup. Why are certain lenses chosen at certain places? How can you manipulate the final image? Use geometrical optics to calculate the size of the final image on the CCD camera. Represent also graphically. How would you change this set-up in order to be able to see the Fourier transform of the object on an extra CCD camera?

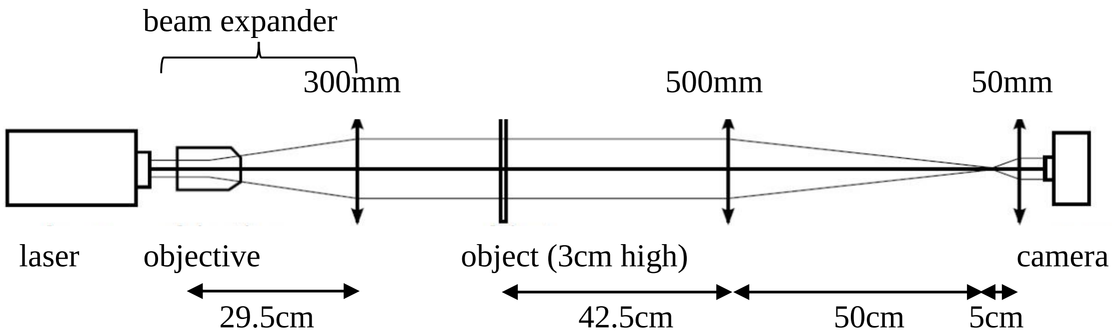

Figure 5: Illustration of the image manipulation setup

- Graphically represent a rectangular function together with its Fourier series, which is truncated after 1, 2, 3, ... terms. What happens to the Fourier series as more and more terms are involved. How can you relate this to image contrast? How could you use this in combination with the setup from the exercise above?

4. Experiment

4.1 Laser alignment

It may very well be that the laser alignment is the most time consuming step of this lab work. Follow these steps to align all components in the most efficient way.

1. Put the Laser and the CCD camera on the rails. Adjust the heights so that it is within the range of the other components. Now fix the laser at the beginning of the rails, and slide the camera back and forth to check that the laser beam reaches the middle of the CCD chip at each position.
2. Put the beam expander so close to the laser that you can see the circles of red laser light on the wall. Adjust the beam expander to center the circles around the same center point, put the CCD onto the rails again and check whether the beam is still parallel to the rails.

3. Put the first lens at a distance of maximum 1 focal distance away from the beam expander. You can use the target caps on the lens to check its correct height. Use a white card to check if the beam is equally broad at each position. Slightly adjust the position of the lens if not. Now move the CCD camera back and forth again and adjust the lens further if needed.
4. Follow the same strategy to put the other lenses in position.
5. If everything is aligned correctly, changing the position of the object in figure 5 should not change its size on the CCD camera; only the sharpness will be influenced.

4.2 Fundamental experiments

In this part you will first look at the Fourier image of a grating and a circular aperture. The setup is as in figure 6. Do you find the Fourier images you expected? For the grating, measure the distances between the maxima and calculate the grating period.

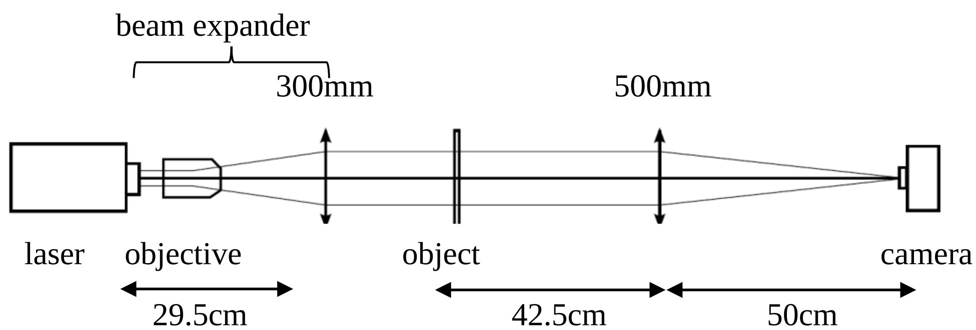

Figure 6: Illustration of the Fourier transform setup.

The basic setup for the next experiments is the one in figure 5. However we want to image the Fourier image at the same time using a beamsplitter behind the second lens.

Place an S-shaped aperture in the object plane. Find out which object you have to use in the Fourier plane in order to get multiple S'es on a vertical, respectively horizontal line as a final image. (Think about the convolution principle.)

4.3 Image manipulation

We will use the same setup as for the multiplication experiment, and two new images in the object plane. The first image is an image of a house built up of lines with different orientations. This can be used to improve your understanding of what is happening. The second image is a photo of a church, which is more complex and realistic than the line house. Make the following images and explain what you see:

- A sharp image without any filtering.
- An image where the horizontal, respectively vertical edges are emphasized.
- An image that looks not sharp. The edges are smoothed out.

- An image where all edges are emphasized.

4.4 Other applications: Text recognition

The setup in figure 5 can also be used to do text recognition. Think about the following situation: You have a piece of text in the object plane, and want to find the location of the o's in this text. What filter do you have to place in the Fourier plane? Why is this technique hard to perform in practice? Does it work equally well with all letters? Why does one want to use the Fourier transform in order to recognize text?

5 Reference

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