Special Problem:
Inventing “New” Mathematics

As Dirac has demonstrated with his $\delta$-function, physicists can invent their own mathematical tools, which may later make their way into mathematics itself. Inventing new mathematics is a creative process which is difficult to teach. Still, this exercise asks you to do exactly this.

As you have learned following Riemann, the integral $\int_a^b f(x)dx$ can be viewed as a continuous limit of a discrete sum $\sum_x f_x$. Imagine that, in order to develop an exciting new idea in theoretical physics, you need to generalize the product $\prod_x f_x$ in a similar way. In other words, you are in urgent need of a mathematical tool that allows you to build the “continuous product” of all values of a non-negative real-valued function $f(x)$ in the interval $x \in [a, b]$. You already have a notation for the desired “product integral”, you denote it by $\mathcal{P}_a^b f(x)^{dx}$, which should have the following properties in analogy to ordinary integration as a proper limit of a discrete sum.

1) As an analog of

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

you demand

$$\mathcal{P}_a^b f(x)^{dx} \cdot \mathcal{P}_b^c f(x)^{dx} = \mathcal{P}_a^c f(x)^{dx}.$$  

2) As a replacement for

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

the product analog should obey

$$\mathcal{P}_a^b f(x)^{dx} = \left[\mathcal{P}_a^b f(x)^{dx}\right]^{-1}.$$  

3) Finally, as the analog of

$$F(b) - F(a) = \int_a^b f(x)dx, \quad \lim_{dx \to 0} \frac{F(x + dx) - F(x)}{dx} = f(x)$$


you demand

\[
\frac{F(b)}{F(a)} = \mathcal{P}_a^b f(x)^{dx}, \quad \lim_{dx \to 0} \left[ \frac{F(x + dx)}{F(x)} \right]^{1/dx} = f(x).
\]

a) Explicitly construct a “product integral” \( \mathcal{P}_a^b f(x)^{dx} \) by expressing it in terms of existing mathematical structures, and show that it indeed has the desired properties 1), 2), and 3). Mathematical rigor is not required, just the right ideas.

b) Use your newly invented mathematical tool to show that in analogy to

\[
\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx
\]

the “product integral” obeys

\[
\mathcal{P}_a^b [f(x)g(x)]^{dx} = \mathcal{P}_a^b f(x)^{dx} \cdot \mathcal{P}_a^b g(x)^{dx}.
\]

c) Show that the analog of \( \int_{-a}^a f(x)dx = 0 \) for an odd function \( f(-x) = -f(x) \) is the relation \( \mathcal{P}_{-a}^a f(x)^{dx} = 1 \) for a “self-reciprocal” function obeying \( f(-x) = f(x)^{-1} \).

d) Evaluate \( \mathcal{P}_0^1 x^{dx} \), in other words, take the “continuous product” of all real numbers between 0 and 1.