85) Zeros of Complex Functions

Consider the following complex functions \( f_n(z) \in \mathbb{C} \) and find all its zeros \( z_0 \in \mathbb{C} \) (such that \( f_n(z_0) = 0 \)).

\[
a) \quad f_1(z) = z^5 - 1 , \\
b) \quad f_2(z) = z^3 + i , \\
c) \quad f_3(z) = \exp(z) , \\
d) \quad f_4(z) = \cosh(z) , \\
e) \quad f_5(z) = \frac{\sinh(z)}{z} .
\]

86) Algebra of Holomorphic Functions

A holomorphic function \( f(z) = u(x, y) + iv(x, y) \in \mathbb{C} \) with \( z = x + iy \) obeys the Cauchy-Riemann differential equations

\[
\frac{\partial u(x, y)}{\partial x} = \frac{\partial v(x, y)}{\partial y} , \quad \frac{\partial u(x, y)}{\partial y} = -\frac{\partial v(x, y)}{\partial x} .
\]

Assume that the functions \( f(z) \) and \( g(z) \) are holomorphic. Show that the following algebraic combinations of the two functions again obey the Cauchy-Riemann differential equations

\[
a) \quad f(z) + g(z) , \\
b) \quad f(z)g(z) , \\
c) \quad \frac{f(z)}{g(z)} , \\
d) \quad f(g(z)) .
\]
87) Analytic Continuation

Assume that the following functions \( u_n(x, y) \) and \( v_n(x, y) \) are real or imaginary parts of analytic functions \( f_n(z) = u_n(x, y) + iv_n(x, y) \) with \( z = x + iy \in \mathbb{C} \). Construct the complete functions \( f_n(z) \) from just their real or imaginary part

\[
\begin{align*}
\text{a)} & \quad u_1(x, y) = x^2 - y^2 , \\
\text{b)} & \quad v_2(x, y) = \exp(-y) \sin(x) , \\
\text{c)} & \quad u_3(x, y) = \exp(x^2 - y^2) \cos(2xy) \\
\text{d)} & \quad v_4(x, y) = 1 .
\end{align*}
\]