Solutions to HW23

76) Integrals with \( \delta \)-Functions

a) We use the Heaviside step function \( \theta(x) = \begin{cases} 0 & x < 0, \\ 1 & x \geq 0 \end{cases} \)

\[
\int_a^b dx \delta(x - 1) = \int_{\mathbb{R}} dx \theta(x - a) \theta(b - x) \delta(x - 1) = \theta(1 - a) \theta(b - 1),
\]

b) For \( \alpha \in \mathbb{R} \setminus \{0\} \), the Dirac delta has the scaling property \( \delta(\alpha x) = \frac{1}{|\alpha|} \delta(x) \)

\[
\int_{\mathbb{R}} dx \delta(ax + b) = \int_{\mathbb{R}} dx \frac{1}{|\alpha|} \delta \left(x + \frac{b}{a}\right) = \frac{1}{|\alpha|},
\]

c) For \( g(x) \) a continuously differentiable function with roots \( x_i \), the composition with the Dirac distribution is defined as \( \delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|} \)

\[
\int_{\mathbb{R}} dx \delta(a^2 - x^2) e^{-bx} = \int_{\mathbb{R}} dx \frac{1}{2|a|} [\delta(x + a) + \delta(x - a)] e^{-bx}
\]
\[
= \frac{1}{2|a|} (e^{ab} + e^{-ab}) = \frac{1}{|a|} \cosh(ab),
\]

Note that the scaling property used in (b) is a special case of the composition of the Dirac distribution with the function \( g(x) = \alpha x \).

d) Using integration by parts we obtain

\[
\int_{\mathbb{R}} dx \delta'(x + a) e^{-bx} = b \int_{\mathbb{R}} dx \delta(x + a) e^{-bx} = be^{ba},
\]

e) We first evaluate the integral over \( y \) making use of one of the two \( \delta \)-functions i.e. we can directly substitute \( y = \pm x \) in the rest of the integrand.
Note that $\delta(-2x) = \delta(2x)$.

\[
\int_{\mathbb{R}} dx \int_{\mathbb{R}} dy \delta(x+y)\delta(x-y)f(x,y) = \int_{\mathbb{R}} dx \delta(2x)f(x,x) = \int_{\mathbb{R}} dx \frac{1}{2}\delta(x)f(x,x) = \frac{1}{2}f(0,0),
\]

f) Recall that $d^3x = dx_1 dx_2 dx_3$ and $\delta(\vec{x} - \vec{y}) = \delta(x_1 - y_1)\delta(x_2 - y_2)\delta(x_3 - y_3)$

\[
\int_{\mathbb{R}^3} d^3x \delta(\vec{x} - \vec{y}) \exp \left[-\frac{1}{2b^2}(\vec{x} - \vec{a})^2\right] = \exp \left[-\frac{1}{2b^2}(\vec{y} - \vec{a})^2\right].
\]

77) Differential Equations with $\delta$-Functions

For the computation of the integrals the results from exercise (76) are used.

a) $f'(x) = \delta(x-a), \quad f(-\infty) = 0$,

\[
\Leftrightarrow \quad \int_{f(-\infty)}^{f(x)} df(y) = \int_{-\infty}^{x} dy \delta(y-a)
\]

\[
\Leftrightarrow \quad f(x) = \int_{\mathbb{R}} dy \theta(x-y)\delta(y-a) = \theta(a-x).
\]

b) $f''(x) = \delta(x-a)e^{-bx}, \quad f(-\infty) = f'(-\infty) = 0$,

\[
\Leftrightarrow \quad f'(x) = \int_{-\infty}^{x} dy \delta(y-a)e^{-by} = \int_{\mathbb{R}} dy \theta(x-y)\delta(y-a)e^{-by} = \theta(a-x)e^{-ba},
\]

\[
\Leftrightarrow \quad f(x) = e^{-ba} \int_{-\infty}^{x} dy \theta(a-y) = e^{-ba} \int_{\mathbb{R}} dy \theta(x-y)\theta(a-y)
\]

\[
= -e^{-ba} \int_{\mathbb{R}} dy [y\theta(x-y)\delta(y-a) - y\theta(a-y)\delta(x-y)]
\]

\[
= -e^{-ba} [a\theta(x-a) - x\theta(a-x)] = -e^{-ba}(a-x)\theta(x-a).
\]
c) \( f''(x) = -f(x) + b\delta(x - a), \quad f(a) = f'(a) = 0. \)

Note that
\[
\begin{align*}
  x < a : & \quad f_1(x) = A_1 \sin(a - x), \\
  x > a : & \quad f_2(x) = A_2 \sin(x - a)
\end{align*}
\]

The function \( f(x) \) itself should be smooth and satisfy \( f(a) = f_1(a) = f_2(a) = 0 \). This is already verified by construction of \( f_1 \) and \( f_2 \). On the other hand, because of the \( \delta \)-function appearing in the differential equation, the first derivative \( f'(x) \) is not continuous at the point \( x = a \)
\[
\lim_{x \to a} [f_2'(x) - f_1'(x)] = b
\]
\[
\Leftrightarrow \quad A_2 + A_1 = b
\]
\[
\Rightarrow \quad f(x) = \sin(a - x) [A_1 - b\theta(x - a)]
\]

The constant \( A_1 \) can now be determined with the constrain
\[
f'(a) = 0 \quad \Leftrightarrow \quad A_1 = b
\]
\[
\Rightarrow \quad f(x) = b \sin(a - x) [1 - \theta(a - x)].
\]

The same result could be obtained by construction of the Green function as showed in the lecture notes.

d) The general solution of the differential equation \( f''(x) = \delta(x) \) is \( f_{A,B}(x) = A + Bx + x\theta(x) \), with \( A, B \in \mathbb{R} \). Note that we can construct an odd and an even solution as following
\[
f_{even,odd} = \frac{1}{2} [x\theta(x) \pm x\theta(-x)].
\]

78) **Green Function for a Differential Equation**

a) Whereas for \( t < 0 \) the equality is trivially satisfied, for \( t > 0 \) we have the differential equation \( \ddot{x}(t) = \frac{F(t)}{m} \). Integrating twice we find the particular
solution
\[ x(t) = \frac{1}{m} \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 F(t_2). \]

b)
\[ x(t) = \frac{1}{m} \int_{\mathbb{R}} dt_1 \theta(t - t_1) \int_{\mathbb{R}} dt_2 \theta(t_1 - t_2) F(t_2) \]
\[ = \frac{1}{m} \int_{\mathbb{R}} dt_1 \theta(t - t_1) \int_{\mathbb{R}} dt_2 \left[ \frac{d}{dt_2} (t_2 \theta(t_1 - t_2) F(t_2)) - t_2 \delta(t_1 - t_2) F(t_2) - t_2 \theta(t_1 - t_2) \dot{F}(t_2) \right] \]
\[ = \frac{1}{m} \int_{\mathbb{R}} dt_1 \theta(t - t_1) [-t_1 F(t_1)]. \]
The Green function has also the form \( G(t - t') = \theta(t - t')(t - t'). \)

c) Calculate the following limit using L'Hôpital's rule
\[ \lim_{\Omega \to 0} \frac{\sin(\Omega(t - t'))}{\Omega} = \lim_{\Omega \to 0} (t - t') \frac{\cos(\Omega(t - t'))}{1} = t - t'. \]
The Green function reduces therefore to the one of eq. (22.31) in the limits \( \gamma \to 0 \) and \( \omega_0 \to 0. \)