Solutions to HW24

79) Function versus Distribution

a) For $f(x) = \theta(x)e^{-x}$ we define the distribution

$$T_f[\varphi] \doteq \int_{-\infty}^{\infty} dx f(x)\varphi(x) = \int_{0}^{\infty} dx e^{-x} \varphi(x) . \quad (1)$$

In order to show that $T_f$ is a regular distribution we just have to show that $f(x)$ is locally integrable (script 23.4, theorem). First, we note that $\lim_{x \to \pm \infty} f(x) \cdot x^n = 0, \ n \in \mathbb{N}$ (this is just applying the theorem of de l’Hôpital $n$-times), so $f$ does not grow at infinity (at all). Second, for $I = (a, b) \subset \mathbb{R}$, we see that

$$\int_{a}^{b} dx f(x) = \int_{a}^{b} dx \theta(x)e^{-x} = \begin{cases} 0, & \text{if } a, b < 0 \\ 1 - e^{-b}, & \text{if } a \leq 0 < b \\ e^{-a} - e^{-b}, & \text{if } a, b > 0 \end{cases} . \quad (2)$$

Therefore $f$ is locally integrable and $T_f$ is a regular distribution.

b) With $\varphi(x) = x$ we find that

$$T_f[\varphi] = T_f[x] = \int_{0}^{\infty} dx e^{-x} \cdot x = \int_{0}^{\infty} dx e^{-x} \cdot x = [xe^{-x}]_0^{\infty} + \int_{0}^{\infty} dx e^{-x} = 1 . \quad (3)$$

c) The derivative of the distribution $T_f'[\varphi] = T_f'[\varphi]$ is given by

$$T_f'[\varphi] = \int_{-\infty}^{\infty} dx f'(x)\varphi(x) = -\int_{-\infty}^{\infty} dx f(x)\varphi'(x) .$$
For \( \varphi(x) = x \) we find that

\[
T_f[x] = -\int_{-\infty}^{\infty} dx f(x)1 = - \int_{0}^{\infty} dx e^{-x} = -1.
\]

(Note: Technically the function \( \varphi(x) = x \) is not in the space \( S \), however this particular distribution can still be evaluated for \( \varphi \).)

80) Principal Value Integral

We use the following results from Section 23.9 (script):

\[
\lim_{\epsilon \to 0^+} \frac{1}{x \pm i\epsilon} = P\left(\frac{1}{x}\right) \mp i\pi \delta(x) \quad \text{(in the distributional sense)} .
\]

\( P(1/x) \) is the principal value integral of \( 1/x \), i.e. for a test function \( \varphi(x) \) this is given by

\[
P\left(\frac{\varphi(x)}{x}\right) = P\int_{-\infty}^{\infty} dx \frac{\varphi(x)}{x} = \lim_{\epsilon \to 0^+} \left[ \int_{-\infty}^{-\epsilon} dx \frac{\varphi(x)}{x} + \int_{\epsilon}^{\infty} dx \frac{\varphi(x)}{x} \right].
\]

\( f_1(x) = \begin{cases} 1 - |x|/3, & \text{if } |x| \leq 3 \\ 0, & \text{if } |x| > 3 \end{cases} \)

\[
\Rightarrow \lim_{\epsilon \to 0^+} \int dx \frac{1}{x + i\epsilon} f_1(x) = P\left(\frac{f_1(x)}{x}\right) - i\pi \int dx \delta(x) f_1(x)
\]

\[
= P\int_{-\infty}^{\infty} dx \frac{f_1(x)}{x} - i\pi f_1(0)
\]

\[
= \lim_{\epsilon \to 0} \left( \int_{-\infty}^{-\epsilon} dx \frac{f_1(x)}{x} + \int_{\epsilon}^{\infty} dx \frac{f_1(x)}{x} \right) - i\pi
\]

\[
= -i\pi , \quad \text{since } f_1 \text{ is even and } 1/x \text{ is odd}
\]
\[ ii) \quad f_2(x) = \begin{cases} 
-1/2 - x/4, & \text{if } |x| \leq 2 \\
0, & \text{if } |x| > 2 \end{cases} \]

\[ \Rightarrow \lim_{\varepsilon \to 0^+} \int \frac{1}{x - i\varepsilon} f_2(x) = P\left( \frac{f_1(x)}{x} \right) + i\pi \int \delta(x)f_2(x) \]

\[ = P \int_{-\infty}^{\infty} \frac{f_2(x)}{x} + i\pi f_2(0) \]

\[ = \lim_{\varepsilon \to 0} \left( \int_{-\varepsilon}^{\varepsilon} \frac{1}{x} \left( -\frac{1/2}{x} - \frac{x/4}{x} \right) + \int_{\varepsilon}^{\infty} \frac{1}{x} \left( -\frac{1/2}{x} - \frac{x/4}{x} \right) \right) - \frac{i\pi}{2} \]

\[ = \lim_{\varepsilon \to 0} \frac{1}{4} \left( \int_{-\varepsilon}^{\varepsilon} + \int_{\varepsilon}^{\infty} \right) - \frac{i\pi}{2} \]

\[ = -1 - \frac{i\pi}{2} . \tag{7} \]

81) Functions in or out of \( S \)

i) \( \phi_1(x) = P_n(x)e^{-x^2} \), \( P_n \) is a polynomial of degree \( n \):

- \( P_n(x) \in C^\infty, \ e^{-x^2} \in C^\infty \implies \phi_1(x) \in C^\infty \).

- Claim: \( \exists C_{m,k} \) such that \( \sup_{x \in \mathbb{R}} \left| \frac{d^m}{dx} \left( P_n(x)e^{-x^2} \right) \right| \leq C_{m,k} \).

Proof. It is quite easy to see that \( \frac{d^m}{dx} P_n(x)e^{-x^2} = \tilde{P}_{n+k}(x)e^{-x^2} \), with \( \tilde{P}_{n+k}(x) \) being a polynomial of degree \( n+k \). Note that \( x^m \cdot \tilde{P}_{n+k}(x)e^{-x^2} \) is finite for finite \( x \), so we just have to show that

\[ \lim_{x \to \pm\infty} x^m \cdot \tilde{P}_{n+k}(x)e^{-x^2} = \tilde{P}_{m+n+k}(x)e^{-x^2} = 0 . \tag{8} \]

Applying the theorem of de l’hôpital \( (m + n + k) \)-times we find that

\[ \lim_{x \to \pm\infty} \frac{\tilde{P}_{m+n+k}(x)}{e^{x^2}} = \ldots = \lim_{x \to \pm\infty} \frac{C}{q_{m+n+k}(x) \cdot e^{x^2}} = 0 , \tag{9} \]

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with \( q_{n+m+k}(x) \) being a polynomial of degree \( n + m + k \) and \( C \) being constant. Hence, \( \phi_1(x) \in S \).

ii) \( \phi_2(x) = e^{x^3-12x^2} \):

- \( \phi_2(x) \in C^\infty \).
- It holds that \( \lim_{x \to \infty} \phi_2(x) = \exp(\lim_{x \to \infty} (x^3 - 12x^2)) = \infty \), and hence \( \phi_2(x) \not\in S \).

iii) \( \phi_3(x) = e^{-|x|} \):

- \( \phi_3(x) \) is not differentiable at \( x = 0 \), and hence \( \phi_3(x) \not\in C^\infty \), which implies that \( \phi_3(x) \not\in S \), even though \( \phi_3(x) \) would satisfy \( \sup_{x \in \mathbb{R}} |x^m \cdot \phi_3^{(k)}(x)| \leq C_{m,k} \).

iv) \( \phi_4(x) = \frac{e^{-x^2}}{1+x^2} \):

- \( \phi_4(x) \) has a pole at \( x = -3 \), i.e. \( \lim_{x \to -3 \pm} |\phi_4(x)| = \infty \), and hence \( \phi_4(x) \) is not bounded in \( \mathbb{R} \), thus \( \phi_4(x) \not\in S \).

v) \( \phi_5(x) = \frac{e^{-x^4}}{1+x^2} \):

- \( e^{-x^4} \in C^\infty \), \( \frac{1}{1+x^2} \in C^\infty \implies \phi_5(x) \in C^\infty \).
- We again claim that \( \exists C_{m,k} \) such that \( \sup_{x \in \mathbb{R}} |x^m \cdot \phi_5^{(k)}(x)| \leq C_{m,k} \).

**Proof.** Similar to i) we note that \( \phi_5^{(k)}(x) = \frac{p(x)e^{-x^4}}{q(x)} \), with \( p(x), q(x) \)
being polynomials. Again this is finite for finite \( x \) and we find that (using de l’hôpital):

\[
\lim_{x \to \pm \infty} x^m \cdot \frac{p(x)}{q(x) e^{x^4}} \leq \ldots \implies \lim_{x \to \pm \infty} \frac{C}{\tilde{q}(x)e^{x^4}} = 0 ,
\]  

(10)

with \( \tilde{q}(x) \) being a polynomial and \( C \) a constant.

vi) \( \phi_6(x) = \theta(x) \):

- \( \phi_6(x) \) is not differentiable at \( x = 0 \). Further, it also holds that \( \lim_{x \to \infty} x \cdot \theta(x) = \infty \), and hence \( \phi_6(x) \notin S \).

R. Moser