Solutions to HW30

97) Integral along a Segment of $\mathbb{C}$

For the evaluation of the integral

$$I = \int_0^\infty \frac{dx}{1 + x^N}, \quad N \in \mathbb{N}_{>1}$$

we first look for the ($N$-many) roots of the complex polynomial function $p(z) = 1 + z^N$ for $N > 1$ a natural number (recall that these are the poles we need for the residue theorem). We parametrize the complex plane as $z = re^{i\phi}, \quad r \in \mathbb{R}, \quad \phi \in [0, 2\pi)$ and for the $k$-th root we obtain

$$p(z_k) = 0 \iff r_k^N e^{iN\phi_k} = -1,$$

$$\iff r_k = 1, \quad iN\phi_k = i(2k + 1)\pi, \quad k \in \mathbb{Z}_N$$

$$\iff z_k = e^{i\pi\frac{2k+1}{N}}, \quad k \in \mathbb{Z}_N.$$ 

Since at the end we want to compute an integral over the positive half real line, we only keep one of the two closest roots, say the positive one $z_{k-1}$ and integrate along the boundaries of the piece of cake with opening angle $\phi_N = \frac{2\pi}{N}$. Let’s parametrize the integration path. Along the first part on the real axis, which we call $\alpha$, the angle is fixed at $\phi = 0$ and the radius change from $r = 0$ to $r = R$. We call $\beta$ the $N$-th section of the circle and realize that there $r = R$ is fixed and the angle grows from $\phi = 0$ to $\phi = \phi_N$. Finally, we close the loop with $\gamma$, where the angle is fixed again at $\phi = \phi_N$ and the radius varies from $r = R$ to $r = 0$

$$J = \lim_{R \to \infty} \oint_{\alpha+\beta+\gamma} dz \frac{1}{1 + z^N}$$

$$= \lim_{R \to \infty} \left\{ \int_0^R dr \frac{1}{1 + r^N} + \int_0^{\phi_N} d\phi \frac{iRe^{i\phi}}{1 + R^N e^{iN\phi}} + \int_R^0 dr \frac{e^{i\phi_N}}{1 + r^N e^{i2\pi}} \right\}$$

$$= (1 - e^{i2\pi/N})I = -2ie^{i\pi/N} \sin\left(\frac{\pi}{N}\right)I.$$
Figure 1: The integration contour for the first four cases.

On the other hand, we can compute the closed integral $J$ using the residue theorem

$$J = 2\pi i \text{Res} \left( \frac{g(z)}{h(z)} \right) = \frac{1}{1 + z^N} ; z_0 = e^{i\pi/N}$$

$$= 2\pi i \left. \frac{g(z)}{h'(z)} \right|_{z = z_0} = \frac{2\pi i}{Ne^{i\pi(N-1)/N}} = -\frac{2\pi ie^{i\pi/N}}{N},$$

$$\Rightarrow -2ie^{i\pi/N} \sin \left( \frac{\pi}{N} \right) I = -\frac{2\pi ie^{i\pi/N}}{N}$$

$$\Leftrightarrow I = \frac{\pi}{N \sin(\pi/N)}.$$ 

We check the result by recalling that we’re already very familiar (MMPH) with this integral for the case $N = 2$

$$I_{N=2} = \int_0^\infty \frac{dx}{1 + x^2} = \arctan(0) - \arctan(\infty) = \frac{\pi}{2} = \frac{\pi}{2 \sin(\pi/2)}.$$
98) Integral along a Cut

Consider the integral

\[ I = \int_{-\infty}^{\infty} dx \frac{\ln(x - i)}{(x + i)^2} \]

and let us first take a deeper look at the integrand. As a complex function, \( f(z) = \frac{\ln(z-i)}{(z+i)^2} \) has a pole at \( z_0 = -i \) and a cut starting at \( z = i \) and extending parallel to the negative real axis.

We close the contour in the upper half-plane and avoid the cut going around it. The loop can be decomposed in six paths. We first define the path \( \alpha \) along the real axis, we then continue upwards on \( \beta \) along a circle with radius \( R \), till we meet close to the cut, say at distance \( \epsilon \). There we first turn right on \( \gamma \), running parallel to the real axis in the positive direction, we than follow \( \delta \) on a half circle of radius \( \epsilon \) around the point \( z = i \) and go back (in negative direction) parallel to the real axis again on \( \eta \). Finally, we close the loop with \( \zeta \), again along the circle with radius \( R \).

The integration on the paths \( \beta \) and \( \zeta \) does not contribute in the limit \( R \to \infty \). Similarly, also the integration on \( \delta \) vanishes for \( \epsilon \to 0 \)

\[
J = \lim_{R \to \infty} \lim_{\epsilon \to 0} \oint = \lim_{R \to \infty} \lim_{\epsilon \to 0} \left( \int_{\alpha} + \int_{\beta} + \int_{\gamma} + \int_{\delta} + \int_{\eta} + \int_{\zeta} \right)
\]

\[
= \lim_{R \to \infty} \left( \int_{\alpha} + \int_{\gamma} + \int_{\eta} \right) = I + \lim_{R \to \infty} \lim_{\epsilon \to 0} \left( \int_{\gamma} + \int_{\eta} \right)
\]

We apply the residue theorem and notice that the function \( f(z) \) has no pole inside the contour. Hence, the function is holomorphic in this region, i.e.
we can find a primary function and any closed integral vanishes

\[ J = 0 \quad \Rightarrow \quad I = - \lim_{R \to \infty} \lim_{\epsilon \to 0} \left( \int_\gamma + \int_\eta \right). \]

We parametrize the paths \( \gamma \) and \( \delta \) with \( z = x + i \epsilon, \ x \in \mathbb{R} \) and note that for small \( \epsilon \) we can approximate \( (x + 2i + i\epsilon)^2 \approx (x + 2i - i\epsilon)^2 \approx (x + 2i)^2 \).

This approximation becomes exact in the limit \( \epsilon \to 0 \)

\[ I = - \lim_{\epsilon \to 0} \left( \int_0^{-\infty} dx \frac{\ln(x + i\epsilon)}{(x + 2i + i\epsilon)^2} + \int_0^{\infty} dx \frac{\ln(x - i\epsilon)}{(x + 2i - i\epsilon)^2} \right) \]

\[ \approx - \lim_{\epsilon \to 0} \int_{-\infty}^{0} dx \frac{\ln(x + i\epsilon) - \ln(x - i\epsilon)}{(x + 2i)^2} \]

\[ = - \lim_{\epsilon \to 0} \int_{-\infty}^{0} dx \frac{\ln(|x + i\epsilon|) - \ln(|x - i\epsilon|) + i \arctan(\epsilon/x) - i \arctan(-\epsilon/x)}{(x + 2i)^2} \]

\[ = - \int_{-\infty}^{0} dx \frac{i\pi - i(-\pi)}{(x + 2i)^2} = -2i\pi \int_{-\infty}^{0} dx \frac{dx}{(x + 2i)^2}. \]

The very important step here is to realize that the complex logarithm is not continuous when passing the cut, therefore we have \( \lim_{\epsilon \to 0} \arctan(\epsilon/x) = \pi \neq -\pi = \lim_{\epsilon \to 0} \arctan(-\epsilon/x) \). Now we are left with a very usual line integral with no singularities

\[ I = -2i\pi \frac{-1}{x + 2i} \bigg|_{x=0}^{x=-\infty} = \pi. \]

99) Green Function of the strongly damped Harmonic Oscillator

a) Use the Fourier transform \( G(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \hat{G}(\omega) e^{i\omega t} \) and compute the time derivatives

\[ \hat{G}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \omega \hat{G}(\omega) e^{i\omega t}, \]

\[ \hat{G}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega (-\omega^2) \hat{G}(\omega) e^{i\omega t}. \]

Inserting these results in the given differential equations one gets a new equation for \( \hat{G}(\omega) \)

\[ (-\omega^2 + i\omega \gamma + \omega_0^2)\hat{G}(\omega) = 1. \quad (1) \]
The right hand side of the equation has been obtained using exercise (83.b).

b) We first solve equation (1) for $\tilde{G}(\omega)$

$$
\tilde{G}(\omega) = \frac{1}{\omega^2_0 + i\omega\gamma - \omega^2} = \frac{1}{[\omega - i(\gamma/2 + \alpha)][\omega - i(\gamma/2 - \alpha)]},
$$

with $\alpha = \sqrt{\gamma^2/2 - \omega_0} \in \mathbb{R}$ real in the strongly damped case $\gamma^2/2 > \omega_0$.

c) $G(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \tilde{G}(\omega)e^{i\omega t}$

$$
= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{e^{i\omega t}}{[\omega - i(\gamma/2 + \alpha)][\omega - i(\gamma/2 - \alpha)]}.
$$

Now it should be decided whether the contour must be closed in the upper or lower half-plane. Respectively, we would parametrize $\omega = Re^{\pm i\phi}$, and thus calculate $e^{i\omega t} = \{\text{complex phase}\} e^{\mp R\sin(\phi)t}$. Since the integrand should vanish in the limit $R \to \infty$, we choose $\omega = Re^{i\phi t}$ (contour $\gamma_+$ in the upper half-plane) in the case $t > 0$ and $\omega = Re^{-i\phi t}$ (contour $\gamma_-$ in the lower half-plane) in the case $t < 0$. Because $\alpha < \gamma/2$, both poles of the integrand are located on the positive imaginary axis. This means that in the lower-half plane the function is holomorphic and any loop integral vanishes. We can
finally apply the residue theorem

\[ G(t) = \frac{1}{2\pi} \left( \oint_{\gamma_+} \theta(t) + \oint_{\gamma_-} \theta(-t) \right) \]

\[ = \frac{1}{2\pi} \left( \oint_{\gamma_+} dz e^{izt} \theta(t) + 0 \right) \]

\[ = 2\pi i \left( \frac{\theta(t)e^{izt}}{z - i(\gamma/2 - \alpha)} \bigg|_{z=i(\gamma/2+\alpha)} + \frac{\theta(t)e^{izt}}{z - i(\gamma/2 + \alpha)} \bigg|_{z=i(\gamma/2-\alpha)} \right) \]

\[ = i\theta(t)e^{-\gamma t/2} \left( \frac{e^{at}}{2i\alpha} - \frac{e^{-at}}{2i\alpha} \right) \]

\[ = \theta(t)e^{-\gamma t/2} \frac{\sinh(\alpha t)}{\alpha} . \]

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