Effective Field Theory

4th Exercise Sheet

5 Decoupling in QED and anomalous magnetic moment

1. Derive the Gordon identity ($q = p_2 - p_1$)

$$\bar{u}(p_2)\,\gamma^\mu\,u(p_1) = \frac{1}{2m}\,\bar{u}(p_2)\,\left[(p_1+p_2)^\mu + i\sigma^{\mu\nu}q_\nu\right]\,u(p_1),$$

by application of the EOM.

2. The contribution of a hypothetical massive photon with propagator

$$\frac{-ig^{\mu\nu}}{q^2 - m_\gamma^2 + i\epsilon} \tag{1}$$

to a_e is given by

$$\Delta a_e = \frac{\alpha}{\pi}\int_0^1 dx \frac{x^2(1-x)}{x^2 + (1-x)\frac{m_\gamma^2}{m_e^2}}.$$

Show that in the limit $m_\gamma \to 0$ this reproduces the Schwinger term $a_e = \alpha/(2\pi)$.

3. Vacuum polarization by a $\mu^+\mu^-$ pair modifies the photon propagator according to

$$\frac{-ig^{\mu\nu}}{q^2+i\epsilon} \to \frac{-ig^{\mu\nu}}{q^2(1-\Pi(q^2))+i\epsilon} = \frac{-ig^{\mu\nu}}{q^2+i\epsilon}(1-\Pi(0)+\Pi(q^2)-\Pi(0)) + \mathcal{O}(\Pi(q^2)^2),$$

where

$$\bar{\Pi}(q^2) \equiv \Pi(q^2) - \Pi(0) = \frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\log\frac{m_\mu^2 - x(1-x)q^2}{m_\mu^2},$$

and we separated $1-\Pi(0)$ because this goes into the renormalization of the charge. We would like to bring the rest into the form (1) to calculate the corresponding contribution to a_e. As a first step, show that

$$\operatorname{Im}\Pi(q^2) = -\frac{\alpha}{3}\sqrt{1-\frac{4m_\mu^2}{q^2}}\left(1+\frac{2m_\mu^2}{q^2}\right).$$

Hint: the imaginary part comes from the logarithm, with $\log(-|a| \pm i\epsilon) = \log|a| \pm i\pi$ *and* $q^2 \to q^2 + i\epsilon$.

4. We can now write $\Pi(q^2)$ as

$$\Pi(q^2) = \Pi(0) + \frac{q^2}{\pi}\int_{4m_\mu^2}^\infty ds \frac{\operatorname{Im}\Pi(s)}{s(s-q^2-i\epsilon)},$$

with the denominator of the desired form (1). Show that

$$\Delta a_e = -\frac{\alpha}{\pi^2}\int_{4m_\mu^2}^\infty ds\,\frac{\operatorname{Im}\Pi(s)}{s}\int_0^1 dx \frac{x^2(1-x)}{x^2+(1-x)\frac{s}{m_e^2}}. \tag{2}$$

5. Now revert the previous trick to find

$$\Delta a_e = \frac{\alpha}{\pi}\int_0^1 dx(1-x)\bar{\Pi}\left(-\frac{x^2 m_e^2}{1-x}\right) = \frac{1}{45}\frac{m_e^2}{m_\mu^2}\left(\frac{\alpha}{\pi}\right)^2 + \mathcal{O}\left(\frac{m_e^4}{m_\mu^4}\right),$$

or, alternatively, directly perform the integral in Eq. (2). We have thus confirmed the result already given in the lecture, with heavy flavors decoupling as $1/m_\mu^2$.