Effective Field Theory

5th Exercise Sheet

6 $SU(N)$ gauge theory

1. Show that the unitarity of the $SU(N)$ matrices entails hermiticity of the generators and that the requirement $\det U = 1$ implies that the generators have to be traceless.

2. Show that the structure constants f_{abc} of $SU(N)$ are real and fulfill the Jacobi identity

$$f^{abd}f^{dce} + f^{bcd}f^{dae} + f^{cad}f^{dbe} = 0$$

   This identity can be obtained by considering the Jacobi identity

$$[[A,B],C] + [[B,C],A] + [[C,A],B] = 0$$

   for the generator matrices $\boldsymbol{T}^a$ and rewriting the commutators in term of structure constants using their defining relation

$$[\boldsymbol{T}^a, \boldsymbol{T}^b] = i f^{abc}\, \boldsymbol{T}^c\,.$$

3. Show that the conjugate representation

$$T^a_{\bar{N}} = -(T^a)^T = -(T^a)^*$$

   and adjoint representation

$$(T^a_A)_{bc} = -i f^{abc}$$

   are indeed representations of $SU(N)$. What is the conjugate representation of the adjoint representation?

4. For a representation $\boldsymbol{T}^a_R$ of a Lie group, the quantity $\boldsymbol{C}_R = \sum_a \boldsymbol{T}^a_R \boldsymbol{T}^a_R$ is called the quadratic Casimir operator of the representation.

   (a) Show that this quantity commutes with all generators $\left[\boldsymbol{C}_R, \boldsymbol{T}^b_R\right] = 0$. For an irreducible representation Schur's lemma then implies that the operator is proportional to the unit matrix $\boldsymbol{C}_R = C_R \mathbf{1}$.

   (b) Compute the values of C_F and C_A, the Casimir invariants of the fundamental and adjoint representation, respectively, i.e.

$$\boldsymbol{T}^a \boldsymbol{T}^a = C_F\, \mathbf{1}\,, \qquad f^{acd} f^{bcd} = C_A \delta^{ab}\,.$$

   Remember that we normalized

$$\mathrm{Tr}\big(\boldsymbol{T}^a \boldsymbol{T}^b\big) = T_F\, \delta^{ab} = \frac{1}{2}\delta^{ab}\,.$$

For C_A, show first that

$$f^{acd}f^{bcd} = 4\mathrm{Tr}\big(C_F \boldsymbol{T}^a\boldsymbol{T}^b - \boldsymbol{T}^a\boldsymbol{T}^c\boldsymbol{T}^b\boldsymbol{T}^c\big)$$

and simplify the last term using

$$\boldsymbol{T}^a_{ij}\boldsymbol{T}^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right),$$

which follows when considering the decomposition of a general $N \times N$ matrix into the unit matrix and $\boldsymbol{T}^a$.

7 "Magic relation" for the anomalous dimension

In this problem we provide arguments for the "magic relation"

$$\gamma = 2\alpha_s \frac{\partial Z_1}{\partial \alpha_s} \tag{1}$$

between the anomalous dimension γ and the first term Z_1 in the ϵ expansion of the Z factor

$$Z = 1 + \sum_{k=1}^{\infty} \frac{1}{\epsilon^k} Z_k(\alpha_s),$$

as it holds in the $\overline{\mathrm{MS}}$ scheme in dimensional regularization.

1. We first need the β function in d dimensions. To this end, use that $\mu\frac{d}{d\mu}\alpha_s^{(0)} = 0$, with bare coupling $\alpha_s^{(0)} = Z_g^2\mu^{2\epsilon}\alpha_s(\mu)$ to show

$$\beta(\alpha_s, \epsilon) = -2\epsilon\alpha_s - 2\alpha_s Z_g^{-1}\mu\frac{d}{d\mu}Z_g.$$

2. Now write $\beta(\alpha_s, \epsilon) = \beta(\alpha_s) + \sum_{k=1}^{\infty}\epsilon^k\beta_k(\alpha_s)$, and use $\mu\frac{d}{d\mu}Z_g = \frac{\partial Z_g}{\partial \alpha_s}\beta(\alpha_s, \epsilon)$ to find

$$Z_g\beta(\alpha_s, \epsilon) = -2\epsilon\alpha_s Z_g - 2\alpha_s\frac{\partial Z_g}{\partial \alpha_s}\beta(\alpha_s, \epsilon).$$

   Expanding this relation at large ϵ, you should find $\beta_1 = -2\alpha_s$, $\beta_k = 0$ for $k > 1$, and the first "magic relation" $\beta(\alpha_s) = 4\alpha_s^2\frac{\partial Z_{1g}}{\partial \alpha_s}$, where Z_{1g} is the first term in the ϵ expansion of Z_g.

3. Finally, repeat the same strategy for the anomalous dimension encountered in the lecture, which should lead to Eq. (1).