Effective Field Theory

6th Exercise Sheet

8 Tree-level $\pi\pi$ scattering and reparameterization invariance

We consider the leading-order $SU(2)$ ChPT Lagrangian

$$\mathcal{L} = \frac{F_\pi^2}{4}\mathrm{Tr}\big(\partial_\mu U \partial^\mu U^\dagger + \chi U^\dagger + \chi^\dagger U\big),$$

where $\chi = 2B\hat{m}\mathbb{1}$ if isospin breaking is neglected (i.e., $m_u = m_d = \hat{m}$). For the matrix U we will consider the parameterizations

$$\begin{aligned} U_\sigma &= \sqrt{1 - \frac{\boldsymbol{\pi}^2}{F_\pi^2}} + \frac{i}{F_\pi}\boldsymbol{\sigma}\cdot\boldsymbol{\pi}, \\ U_{\exp} &= \exp\left(\frac{i}{F_\pi}\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right), \\ U_g &= \exp\left(\frac{i}{F_\pi}\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\, g\Big(\frac{\boldsymbol{\pi}^2}{F_\pi^2}\Big)\right), \end{aligned}$$

where $g(x)$ is an arbitrary real, smooth function with $g(0) = 1$.

1. Show that U_σ, $U_{\exp}$, U_g are indeed admissible parameterizations of the pion fields, which all lead to $M_\pi^2 = 2\hat{m}B$.
   Hint: it is easiest to work with the Cartesian components of the pion fields π^a for now, instead of the combinations with definite electric charge.

2. The scattering amplitude for $\pi^a(p_a)\pi^b(p_b) \to \pi^c(p_c)\pi^d(p_d)$ is given by

   $$T^{ab,cd}(s,t,u) = \delta^{ab}\delta^{cd}A(s,t,u) + \delta^{ac}\delta^{bd}A(t,u,s) + \delta^{ad}\delta^{bc}A(u,s,t),$$

   with Mandelstam variables s, t, u. Calculate the tree-level expression for $A(s,t,u)$ for the three parameterizations of the pion fields.

3. Calculate the scattering amplitudes with total isospin $I = 0, 1, 2$.
   Hint: for this part it helps to go to the particle basis.

4. Evaluate the amplitudes at threshold and obtain the S-wave scattering lengths with total isospin $I = 0, 2$.