Critical Phenomena - Exercise Set 1

1) Ising 1D Exact Solution

In this question we will find the exact solution of the Ising model in one dimension. This is a simple but illustrative example. Consider N spins $s_i = \pm 1$ in one dimension with periodic boundary conditions, i.e. $s_{N+1} = s_1$ and Hamiltonian

$$H = -J \sum_i s_i s_{i+1} - \frac{B}{2} \sum_i (s_i + s_{i+1}) \tag{1}$$

The partition function of the system is given by

$$Z = \sum_{s_1 = \pm 1} \cdots \sum_{s_N = \pm 1} e^{-\beta H} \tag{2}$$

where β is the inverse temperature.

a) Show that the partition function may be written as $Z = \mathrm{tr}(T^N)$ where the *transfer matrix* T is a 2×2 matrix. Find T.

b) Find the eigenvalues of T and therefore an explicit expression for Z.

c) Compute the magnetization $m = \frac{1}{N} \langle \sum_i s_i \rangle$ in the thermodynamic limit, and show that it is zero for $B = 0$. This implies that the system has no phase transition at any finite temperature. What happens for $B \neq 0$?

d) Consider the *correlation function* $C(i, j) = \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle$. Explain why $C(i, j)$ depends only on the difference $i - j$ and compute explicitly $C(i, 0)$ for $B = 0$. *Hint: write the expression for $\langle s_i s_j \rangle$ and repeat the steps leading to the transfer matrix in part a). You can write $\langle s_i s_j \rangle$ in terms of the transfer matrix, but two new matrices are required.*

e) In general, $C(i - j) \sim e^{-|i-j|/\xi} / |i - j|^{d-2+\eta}$ where ξ is the *correlation length* and η is a critical exponent. Find ξ and η. For which values of the parameters does $\xi \to \infty$?

2) Ising 1D Mean Field Theory

As we've just seen, the 1D Ising model can be solved exactly. However, this is not true for more complicated models. A technique which is often used in more complicated cases is *mean field theory*, and here we will see how it works for the 1D Ising model. Consider the same setup as in Exercise 1. Now write the interaction term in the Hamiltonian as

$$s_i s_j = (s_i - m + m)(s_j - m + m) = m^2 + m(s_i - m) + m(s_j - m) + (s_i - m)(s_j - m) \tag{3}$$

where m is the equilibrium magnetization. Under the assumption that the deviations from equilibrium are small, the last term in (3) may be neglected.

a) Under this approximation, compute explicitly the partition function Z.

b) Use this partition function to show that the equilibrium magnetization obeys the self-consistency equation $m = \tanh(\beta B + \beta J q m)$, where q is the number of nearest neighbour pairs.

c) In the case $B = 0$ draw an appropriate graph to show that there is a critical value β_c at which the solutions m of the self-consistency equation change abruptly. How does this compare with the exact solution of Exercise 1?

d) In the case $B \neq 0$ find the asymptotic form of the solution in the infinite temperature limit $\beta \to 0$.

e) The critical exponent $\widetilde{\beta}$ is defined by the equation $m(T) = (T_c - T)^{\widetilde{\beta}}$ in the vicinity of T_c, where $T = 1/\beta$ and $T_c = 1/\beta_c$. Set $B = 0$ and find $\widetilde{\beta}$ by expanding the self-consistency equation in a series. *Note: $\widetilde{\beta}$ is usually called β, but here we added a $\sim$ in order to avoid confusion.*

3) Ginzburg-Landau Theory

The Ising model has a $\mathbb{Z}_2$ global symmetry given by flipping all the spins at once, $s_i \to -s_i$. Under this flip, the magnetization changes sign $m \to -m$. In the spirit of Ginzburg-Landau theory we may write an *effective free energy*

$$f(m,T) = a(T) + \frac{1}{2}b(T)m^2 + \frac{1}{4}c(T)m^4 + \cdots \tag{4}$$

by including the lowest order terms which respect the symmetry $m \to -m$, while the coefficients depend on the temperature T.

a) What conditions must the coefficients satisfy in order for f to be bounded below?

b) Suppose that the system undergoes a phase transition at $T = T_c$. What conditions must $b(T)$ satisfy in order for this to occur?

c) Expand $b(T)$ to its lowest order, and therefore find the minimum m of the free energy in the cases $T < T_c$ and $T > T_c$.

d) Find the critical exponent $\widetilde{\beta}$ as defined in Exercise 2. Does the answer agree with the result of Exercise 2?

Critical Phenomena - Exercise Set 2

4) Tricritical point

This exercise is similar in spirit to Exercise 3, but a little more involved. Consider the free energy

$$f(m, T) = a(T)m^2 + b\,m^4 + c\,m^6 \tag{5}$$

where $c > 0$ and b, c do not depend on the temperature.

a) Suppose $b \neq 0$. What conditions must the coefficients satisfy in order for there to be a phase transition? Suppose that there indeed is a phase transition. What is its order, and how does the magnetization behave at the transition?

b) Suppose $b = 0$. At what value of a does one have a phase transition? This is called a *tricritical point*. Compute the mean-field critical exponents $\alpha, \beta, \gamma, \delta$. *Hint: for the last two, you need to add a magnetic field term $-Bm$ to the free energy*

5) Asymptotic behaviour of the Green's function

While performing calculations one often encounters the Green's function $G(\mathbf{x})$ for the operator $\nabla^2 + \mu^2$ in d dimensions, which is given in momentum space by

$$G(\mathbf{x}) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{k^2 + \mu^2} \tag{6}$$

In this exercise, we will use the *saddle-point expansion* to evaluate its asymptotic behaviour.

a) Explain why $G(\mathbf{x})$ depends only on $r = |\mathbf{x}|$. For which A is the identity $\frac{1}{A} = \int_0^\infty dt\, e^{-tA}$ valid? Use it to write $G(r) \sim \int_0^\infty e^{-S(t)}$ for some S, ignoring a prefactor.

b) Find the minimum of $S(t)$. Expanding $S(t)$ to second order in t around the minimum and performing the corresponding integral (this is the saddle-point approximation), find the asymptotic behaviour of $G(r)$ in the two regimes $r \ll 1/\mu$ and $r \gg 1/\mu$.

6) Superfluid order parameter

Consider the following free energy, which describes an idealized superfluid:

$$F = \int dx \left(a(T)\,|\psi|^2 + b\,|\psi|^4 - \gamma \left|\frac{d\psi}{dx}\right|^2 + \mu \left|\frac{d\psi}{dx}\right|^4 \right) \tag{7}$$

where $\psi(x)$, the *superfluid order parameter*, is a complex function and $b, \gamma, \mu > 0$.

a) Consider the case where ψ is space-independent. What are the possible ground states? Show that there is a phase transition at $a = 0$ and compute its mean field exponents. Does this involve spontaneous symmetry breaking?

b) Write the free energy in momentum space.

c) Suppose that only one Fourier mode contributes to the momentum-space free energy. What value of the momentum minimises F? Show that the system undergoes a phase transition and find the critical value of a. What symmetries are broken?

7) Wick's identity

In this exercise we prove a useful formula valid for Gaussian theories, which we'll need later. Suppose that we have N real variables ϕ_i and let $\phi = (\phi_1, \ldots, \phi_N)$. For any function $f(\phi)$, define

$$\langle f(\phi) \rangle = \frac{1}{Z} \int_{-\infty}^{+\infty} d^N\phi \, \exp\left[-\frac{1}{2} \phi \cdot G^{-1} \cdot \phi \right] \tag{8}$$

where G is an invertible $N \times N$ matrix.

a) What must be the value of Z if we require that $\langle 1 \rangle = 1$?

b) Show that $\langle \phi_i \phi_j \rangle = G_{ij}$. *Hint: Add a term* $J \cdot \phi \equiv \sum_i J_i \phi_i$ *to the action, differentiate appropriately and then set* $J = 0$.

c) Therefore show that for any constant vector A, we have

$$\left\langle \exp\left(\sum_i A_i \phi_i \right) \right\rangle = \exp\left(\frac{1}{2} \sum_{ij} A_i A_j \langle \phi_i \phi_j \rangle \right)$$

Critical Phenomena - Exercise Set 3

8) XY Model

The XY model is used to describe several condensed matter systems, and consists of two-components spins $\mathbf{S} = (S_x, S_y)$ with $\mathbf{S}^2 = 1$ placed on the sites of a d-dimensional periodic hypercubic lattice. The Hamiltonian is

$$H = -K \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j \tag{9}$$

with $K > 0$, so that the spins want to be aligned. The notation $\langle ij \rangle$ means that i and j are nearest neighbours.

a) What is the symmetry of this system?

b) Express each spin $\mathbf{S}_i$ in terms of an angle θ_i and write the partition function $Z = \int \prod_i d\mathbf{S}_i \, e^{-\beta H}$ in terms of these angles.

c) At low temperature (large β) the spins vary slowly from site to site. Thus we may expand H quadratically in the angles θ_i. What is the resulting low-temperature Hamiltonian?

d) Use a Fourier transformation of the angle variables to diagonalize the low-temperature Hamiltonian.

e) Using Exercise 7, find an expression for $\langle \mathbf{S}_0 \cdot \mathbf{S}_\mathbf{x} \rangle = \mathrm{Re} \langle \exp\left[i(\theta_\mathbf{x} - \theta_0) \right] \rangle$ in the form of an integral. Use the stationary phase approximation to show that for large $|\mathbf{x}|$, the integral is dominated by small momenta. Hence find the asymptotic behaviour of the correlation function $\langle \mathbf{S}_0 \cdot \mathbf{S}_\mathbf{x} \rangle$, depending on the dimension d.

9) A surface model

An approximately flat surface (i.e. with no 'overhangs') in d-dimensions can be described by its height $h(x)$ as a function of the remaining $(d-1)$ coordinates $x = (x_1, \ldots, x_{d-1})$. If the surface has a tension σ, the Hamiltonian is simply $H = \sigma A$ where A is the area of the surface,

$$A = \int d^{(d-1)} x \, \sqrt{1 + (\nabla h)^2} \tag{10}$$

a At suffiently low temperatures there will be only slow variations in h. By expanding the Hamiltonian to quadratic order in h, express the total partition function as a functional integral.

b By using a Fourier transformation to diagonize the quadratic Hamiltonian into normal modes (known as capillary waves), show that the low-energy excitations are described by Goldstone modes. Identify the symmetry breaking responsible for the generation of the Goldstone modes.

c Obtain an expression for the correlation function of the height $\langle(h((x) - h(0))^2\rangle$ in the form of an integral and, without evaluating it explicitly, comment on the form of your result in dimensions $d = 2, 3$ and 4.

10) A yet-to-be-determined question

Critical Phenomena - Exercise Set 4

11) Ising 1D Block spin transformation

In this exercise we will learn how to perform a *block-spin transformation* for the 1D Ising model, which is one of the ways of performing a renormalization group transformation. Consider an infinite chain of spins $s_i = \pm 1$ in one dimension with Hamiltonian

$$H = -J \sum_i s_i s_{i+1} - \frac{B}{2} \sum_i (s_i + s_{i+1}) + C \tag{11}$$

It will become clear later why we introduced the constant term. Define the partition function as

$$Z = \sum_{s_1=\pm 1} \sum_{s_2=\pm 1} \cdots e^{-H} \tag{12}$$

where we set $\beta = 1$ for simplicity. The idea is to "integrate out" half the spins in the above formula, thus leaving Z unchanged, but expressed in terms of a new Hamiltonian H' which is defined in terms of the remaining spins only.

a) Separate the spins into even and odd and perform explicitly the sum over each odd spin in eq.(12). Show that after "integrating out" all the odd spins the partition function takes the same form as in eq.(12), where now the sum runs over all the even spins and the new Hamiltonian H' is identical to H but with new parameters J', B' and C'. *Hint: consider one single odd spin s and its even nearest neighbours s_- and s_+. Write explicitly all the terms in the Hamiltonian where s appears and perform the sum over $s = \pm 1$. Then consider all the possible values that $s_- + s_+$ may take.*

b) Remarkably, after this complicated procedure we have obtained the same Hamiltonian only with different parameters J', B' and C', defined in terms of the original J, B and C. We may then imagine repeating this procedure multiple times, thus "flowing" in the space of parameters. In the case that $B = 0$, check that no new B term is generated by the flow. Therefore, in this case, find the fixed points of the block-spin transformation. How does the correlation length behave at these fixed points?

12) The Lifshitz point

Certain materials, such as liquid crystals, are anisotropic and therefore behave differently in different directions. Consider a d-dimensional system with coordinates $\mathbf{x} = (x, \vec{y})$ where $\vec{y}$ is a $(d-1)$-dimensional vector, and Hamiltonian

$$H = \int dx\, d^{(d-1)}\vec{y} \left[\frac{1}{2}(\partial_x \phi)^2 + \frac{1}{2}(\nabla^2 \phi)^2 + \frac{1}{2} t \phi^2 \right] \tag{13}$$

where ∇ refers to $\vec{y}$ only. Note that the Hamiltonian is quadratic in the x derivative, but quartic in the y derivatives.

1. Write the Hamiltonian in Fourier space.

2. Writing $\mathbf{k} = (k, \vec{q})$ for the momenta, set up a renormalization group transformation such that $k' = \zeta k$, $\vec{q}' = \zeta^a \vec{q}$ and $\phi'(\mathbf{k}') = \zeta^b \phi(\mathbf{k})$ and choose a and b so that the normalization of the derivative terms is unchanged. How does t transform under the renormalization group?

3. Compute $\langle \phi(\mathbf{k})\phi(\mathbf{k}')\rangle$ and the corresponding susceptibility $\chi(\mathbf{k}) = \langle |\phi(\mathbf{k})|^2 \rangle$.

4. What is the scaling dimension of the field ϕ near the Gaussian fixed point? Consider a generic interaction term of the form $g_n \phi^n$. What is the dimension of g_n? Show that the quartic interaction ϕ^4 is relevant in $d < 7$ and irrelevant in $d > 7$.

5. Now add a quartic perturbation to the Hamiltonian, $g\phi^4$. What is its expression in momentum space? Find an integral expression for the susceptibility χ to first order in g. How would you find where the transition takes place from χ?

13) O(N) model

Consider N scalars $\vec{\phi} = (\phi_1, \ldots, \phi_N)$ with Hamiltonian

$$H = \int d^d x \left[\frac{1}{2}(\nabla \vec{\phi})^2 + \frac{1}{2} t \vec{\phi}^2 + g \vec{\phi}^4 \right] \tag{14}$$

which is symmetric under O(N) transformations of the N fields. One can compute the renormalization group beta functions in $d = 4 - \epsilon$ dimensions, which are given by:

$$\frac{dt}{ds} = 2t + \frac{N+2}{2\pi^2} \frac{\Lambda^4}{\Lambda^2 + t} \widetilde{g} \tag{15}$$

$$\frac{d\widetilde{g}}{ds} = \epsilon \widetilde{g} - \frac{N+8}{2\pi^2} \frac{\Lambda^4}{(\Lambda^2 + t)^2} \widetilde{g}^2 \tag{16}$$

where Λ is the cutoff, $\widetilde{g} = \Lambda^{-\epsilon} g$, and the RG transformation is such that $x' = e^s x$.

a) Find the fixed points of the RG flow and show that there is a non-trivial fixed point. This is called the *Wilson-Fisher fixed point.*

b) Linearize the beta functions and study the relevant and irrelevant directions at the Wilson-Fisher fixed point.

c) Compute the critical exponent ν at the Wilson-Fisher fixed point. Assuming that $\eta \sim \epsilon^2$, compute also $\alpha, \beta, \gamma, \delta$ to leading order in ϵ.

d) From the first beta function, we see that t is of order ϵ along the flow from the Gaussian to the Wilson-Fisher fixed point. Under this assumption, find the beta function for $\widetilde{g}$ to leading order in ϵ and solve it.

Critical Phenomena - Exercise Set 5

14) O(N) perturbation

Consider the O(N) model of the previous question with the addition of a term which breaks the O(N) symmetry, $\lambda \sum_{i=1}^{N} \phi_i^4$. To leading order in ϵ, the coupled beta functions for this model are in $d = 4 - \epsilon$,

$$\frac{d\widetilde{\lambda}}{ds} = \epsilon\widetilde{\lambda} - \frac{9}{2\pi^2}\widetilde{\lambda}^2 - \frac{6}{\pi^2}\widetilde{g}\widetilde{\lambda} \tag{17}$$

$$\frac{d\widetilde{g}}{ds} = \epsilon\widetilde{g} - \frac{N+8}{2\pi^2}\widetilde{g}^2 - \frac{3}{\pi^2}\widetilde{g}\widetilde{\lambda} \tag{18}$$

where $\widetilde{g} = \Lambda^{-\epsilon} g$ and $\widetilde{\lambda} = \Lambda^{-\epsilon}\lambda$.

a) Show that the Hamiltonian is positive definite only if $g + \lambda > 0$.

b) Show that the beta functions have four fixed points: the Gaussian fixed point, the Heisenberg fixed point (with $\widetilde{\lambda} = 0$), the Ising fixed point (with (with $\widetilde{g} = 0$)) and the cubic fixed point (with $\widetilde{\lambda}, \widetilde{g} \neq 0$). For which of these fixed points is the energy positive definite?

c) Determine the stability of these fixed points in the (g, λ) plane and plot the RG flow between them. Does the answer depend on N? *Hint: the sign of the eigenvalues of a* 2×2 *matrix can be determined from its determinant and trace.*

15) A fluctuating membrane

A nearly flat two-dimensional membrane embedded in three-dimensional space is described by the Hamiltonian

$$H = \int d^2x \left[\frac{r_0}{2}(\nabla h)^2 + \frac{\kappa_0}{2}\left(1 + (\nabla h)^2\right)^{-5/2}\left(\nabla^2 h\right)^2\right] \tag{19}$$

where h is the height of the membrane, and the second term is related to the curvature of the membrane. Here r_0 is the interface tension and κ_0 is the bending modulus.

a) Assume that h is small so that the square root may be expanded in a series and all terms quartic or higher in h may be neglected. Find integral expressions for $\langle (h(\mathbf{x}) - h(0))^2 \rangle$ and $\langle (\nabla h(\mathbf{x}) - \nabla h(0))^2 \rangle$ and estimate their long-distance behaviour. What does this tell us about the long-distance order of the membrane?

b) Now again expand the square root and keep the first two terms, so that the Hamiltonian now consists of three terms. Let H_0 consist of the two quadratic terms in h. Then split h into long-distance and short-distance modes, $h = h^- + h^+$, where h^+ has support for momenta $\Lambda/\zeta < k < \Lambda$. Using $e^{-H_0[h^+]}$ as a probability distribution, show that

$$\langle h^+(\mathbf{k}) h^+(\mathbf{k}') \rangle_+ = \frac{1}{r_0 k^2 + \kappa_0 k^4}(2\pi^2)\delta(\mathbf{k} + \mathbf{k}')$$

c) Write the quartic term in the Hamiltonian in momentum space, and identify the term which will renormalise the interaction $(\nabla^2 h)^2$. Focus only on this term (so, in particular, ignore field renormalization) and derive the beta function

$$\frac{dk}{ds} \approx -\frac{5}{4\pi}$$

where $\zeta = e^s$. What does this imply for the membrane? Do short distance fluctuations make it more or less flexible on long-distance scales?

16) Sine-Gordon theory in 2D

The Sine-Gordon theory in 2D is related to the BKT transition which we considered during the lectures. In this question we will consider its renormalization. We have a scalar field ϕ with Hamiltonian

$$H = \int d^2x \left[\frac{1}{2}(\nabla\phi)^2 - \lambda_0 \cos(\beta_0\phi)\right] \tag{20}$$

a) What are the naive dimensions of λ_0 and β_0?

b) Decompose the Fourier modes of the field as $\phi(\mathbf{k}) = \phi(\mathbf{k})^- + \phi(\mathbf{k})^+$ where ϕ^+ has support for momenta $\Lambda/\zeta < k < \Lambda$. Show that after integrating out ϕ^+, the Hamiltonian for $\phi^-(\mathbf{x})$ (in real space) becomes

$$H'[\phi^-] = \int d^2x \left[\frac{1}{2}\left(\nabla\phi^-\right)^2 - \lambda_0\langle\cos(\beta_0(\phi^- + \phi^+))\rangle_+\right]$$

to leading order in λ_0. What does $\langle\cdot\rangle_+$ mean here?

c) Show that

$$\langle\phi^+(\mathbf{x})\phi^+(\mathbf{y})\rangle_+ = \int_{\Lambda/\zeta}^{\Lambda} \frac{d^2k}{(2\pi)^2}\frac{e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{k^2}$$

d) Hence, expressing the cosine in terms of exponentials and using Wick's identity from Exercise 7 show that

$$H'[\phi^-] = \int d^2x \left[\frac{1}{2}\left(\nabla\phi^-\right)^2 - \lambda_0\zeta^{-\beta_0^2/4\pi^2}\cos(\beta_0\phi^-)\right]$$

Therefore show that the cosine term is relevant for $\beta_0^2 < 8\pi$ and irrelevant for $\beta_0^2 > 8\pi$.