Critical Phenomena - Exercise Set 2

3) Ising 2D Self-duality

The Ising model in 2D can be solved exactly, but the solution is notoriously difficult. In this exercise we will use a *duality transformation* to find the transition temperature exactly without solving the model. In general, dualities are a powerful tool which may be applied to a wide range of systems. Consider the Ising model on an infinite two-dimensional square lattice,

$$H = -J \sum_{\langle ij \rangle} s_i s_j \qquad\qquad Z(\beta) = \sum_{\{s\}} e^{-\beta H} \tag{4}$$

where $s_i = \pm 1$ and $\langle ij \rangle$ denotes nearest neighbours. The sum in the partition function is over all possible spin configurations. Since J always appears in the product βJ, we can set $J = 1$ and absorb it into β.

a) Show that $e^{\beta s_i s_j} = \cosh \beta + s_i s_j \sinh \beta$. Therefore find f_0 and f_1 such that the partition function may be written as

$$Z(\beta) = \sum_{\{s\}} \prod_{\langle ij \rangle} \sum_{k=0}^{1} f_k(\beta)(s_i s_j)^k$$

b) Show that we can change the order of the product and the sum so that

$$Z(\beta) = \sum_{\{s\}} \sum_{\{k\}} \prod_{\langle ij \rangle} f_{k_{\langle ij \rangle}}(\beta)(s_i s_j)^{k_{\langle ij \rangle}}$$

where $\{k\}$ is a possible assignment of $k_{\langle ij \rangle} = 0, 1$ to each nearest neighbour pair $\langle ij \rangle$.

c) Manipulate the last expression in the partition function in order to show that

$$Z(\beta) = \sum_{\{k\}} \left(\prod_{\langle ij \rangle} f_{k_{\langle ij \rangle}}(\beta) \right) \prod_i \left(\sum_{s_i = \pm 1} s_i^{\sum_{\langle ij \rangle} k_{\langle ij \rangle}} \right)$$

Show that the last term within brackets is non-zero only if $\sum_{\langle ij \rangle} k_{\langle ij \rangle}$ is even.

d) Now we rewrite the partition function in terms on new variables defined on the *dual lattice*. This is again a square lattice, where the sites are shifted by half a lattice spacing in the two directions. Draw the dual lattice in relation to the original lattice and convince yourself that each site of the original lattice corresponds to a unique plaquette in the dual lattice, and viceversa, and that moreover each link of the original lattice uniquely identifies a link of the dual lattice.

e) Define new Ising variables $\sigma_{\tilde{i}} = \pm 1$ on each site $\tilde{i}$ of the *dual* lattice. Show that each $\langle ij \rangle$ identifies a pair $\langle \tilde{i}\tilde{j} \rangle$ of nearest neighbour sites $\tilde{i}$ and $\tilde{j}$ in the dual lattice. Now we set $k_{\langle ij \rangle} = \frac{1}{2}(1 - \sigma_{\tilde{i}} \sigma_{\tilde{j}})$. Is this relation well-defined? Show that this choice makes $\sum_{\langle ij \rangle} k_{\langle ij \rangle}$ always even. It can be shown that this choice is *necessary* in order to make the sum even.

f) Therefore show that in terms of the new variables,

$$Z(\beta) = 2^N \frac{1}{2} \sum_{\{\sigma\}} \left(\prod_{\langle \tilde{i}\tilde{j} \rangle} f_{\frac{1}{2}(1-\sigma_{\tilde{i}}\sigma_{\tilde{j}})}(\beta) \right)$$

where N is the number of spins. *Hint for the factor of* $1/2$*: what happens to the* k *if we send all* $\sigma \to -\sigma$*?*

g) Show that $f_k(\beta) = \cosh(\beta) \exp[k \log \tanh(\beta)]$ for the relevant values of k, and that therefore

$$Z(\beta) = \frac{(\sinh(2\beta))^N}{2} Z(\beta^*)$$

for a value β^* which you will need to find. Show that large (small) β corresponds to small (large) β^*. This shows that the 1D Ising model is *self-dual*: we have an exact relation between the high-temperature and low-temperature partition function.

h) What does the relation for the partition function at point g) mean for the free energy per spin in the thermodyanamic limit? Assuming that there is a unique critical point where the free energy is non-analytic, find the critical value β_c.

4) Ginzburg-Landau Theory

The Ising model has a $\mathbb{Z}_2$ global symmetry given by flipping all the spins at once, $s_i \to -s_i$. Under this flip, the magnetization changes sign $m \to -m$. In the spirit of Ginzburg-Landau theory we may write an *effective free energy*

$$f(m, T) = a(T) + \frac{1}{2}b(T)m^2 + \frac{1}{4}c(T)m^4 + \cdots \tag{5}$$

by including the lowest order terms which respect the symmetry $m \to -m$, while the coefficients depend on the temperature T.

a) What conditions must the coefficients satisfy in order for f to be bounded below?

b) Suppose that the system undergoes a phase transition at $T = T_c$. What conditions must $b(T)$ satisfy in order for this to occur?

c) Expand $b(T)$ to its lowest order, and therefore find the minimum m of the free energy in the cases $T < T_c$ and $T > T_c$.

d) Find the critical exponent $\widetilde{\beta}$ as defined in Exercise 2. Does the answer agree with the result of Exercise 2?

5) Tricritical point

This exercise is similar in spirit to Exercise 3, but a little more involved. Consider the free energy

$$f(m, T) = a(T)m^2 + b\,m^4 + c\,m^6 \tag{6}$$

where $c > 0$ and b, c do not depend on the temperature.

a) Suppose $b \neq 0$. What conditions must the coefficients satisfy in order for there to be a phase transition? Suppose that there indeed is a phase transition. What is its order, and how does the magnetization behave at the transition?

b) Suppose $b = 0$. At what value of a does one have a phase transition? This is called a *tricritical point*. Compute the mean-field critical exponents $\alpha, \beta, \gamma, \delta$. *Hint: for the last two, you need to add a magnetic field term $-Bm$ to the free energy*