Critical Phenomena - Exercise Set 3

6) Superfluid order parameter

Consider the following free energy, which describes an idealized superfluid:

$$F = \int dx \left(a(T) \left|\psi\right|^2 + b \left|\psi\right|^4 - \gamma \left|\frac{d\psi}{dx}\right|^2 + \mu \left|\frac{d\psi}{dx}\right|^4 \right) \tag{7}$$

where $\psi(x)$, the *superfluid order parameter*, is a complex function and $b, \gamma, \mu > 0$.

a) Consider the case where ψ is space-independent. What are the possible ground states? Show that there is a phase transition at $a = 0$ and compute its mean field exponents. Does this involve spontaneous symmetry breaking?

b) Write the free energy in momentum space.

c) Suppose that only one Fourier mode contributes to the momentum-space free energy. What value of the momentum minimises F? Show that the system undergoes a phase transition and find the critical value of a. What symmetries are broken?

7) Wick's identity

In this exercise we prove a useful formula valid for Gaussian theories. Suppose that we have N real variables ϕ_i and let $\phi = (\phi_1, \dots, \phi_N)$. For any function $f(\phi)$, define

$$\langle f(\phi) \rangle = \frac{1}{Z} \int_{-\infty}^{+\infty} d^N\phi \, f(\phi) \, \exp\left[-\frac{1}{2} \phi \cdot G^{-1} \cdot \phi \right] \tag{8}$$

where G is an invertible $N \times N$ matrix.

a) What must be the value of Z if we require that $\langle 1 \rangle = 1$? *Note: you should compute the integral explicitly.*

b) Show that $\langle \phi_i \phi_j \rangle = G_{ij}$. *Hint: Add a term* $J \cdot \phi \equiv \sum_i J_i \phi_i$ *to the action, differentiate appropriately and then set* $J = 0$.

c) Therefore show that for any constant vector A, we have

$$\left\langle \exp\left(\sum_i A_i \phi_i \right) \right\rangle = \exp\left(\frac{1}{2} \sum_{ij} A_i A_j \langle \phi_i \phi_j \rangle \right)$$

8) Asymptotic behaviour of the Green's function

While performing calculations one often encounters the Green's function $G(\mathbf{x})$ for the operator $\nabla^2 + 1/\xi^2$ in d dimensions, which is given in momentum space by

$$G(\mathbf{x}) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{k^2 + 1/\xi^2} \tag{9}$$

where ξ is the correlation length. In this exercise, we will use the *saddle-point expansion* to evaluate its asymptotic behaviour.

a) Explain why $G(\mathbf{x})$ depends only on $r = |\mathbf{x}|$. For which A is the identity $\frac{1}{A} = \int_0^\infty dt\, e^{-tA}$ valid? Use it to write $G(r) \sim \int_0^\infty e^{-S(t)}$ for some S, ignoring a prefactor.

b) Find the minimum of $S(t)$. Expanding $S(t)$ to second order in t around the minimum and performing the corresponding integral (this is the saddle-point approximation), find the asymptotic behaviour of $G(r)$ in the two regimes $r \ll \xi$ and $r \gg \xi$.

9) More asymptotic behaviour

In class we computed the fluctuation corrections to the mean-field approximation, and in doing so we encountered the integral

$$I_d = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + 1/\xi^2)^2} \tag{10}$$

In this question we investigate the behaviour of I_d in different dimensions.

a Perform the angular integrals explicitly in terms of the area S_d of the d-dimensional sphere. What are the problematic regions of the resulting expression for I_d, which could give rise to a divergence?

b Show that the integral diverges for $d > 4$. Introduce a momentum cutoff and find the asymptotic behaviour of the integral in terms of the cutoff.

c Show that the integral is finite for $d < 4$. Perform a change of variables to extract the dimensional dependence of I_d.

This calculation shows that for $d > 4$ the fluctuation corrections are constant on each side of the transition, so they lead at most to a finite jump; on the other hand for $d < 4$, since ξ itself diverges at the transition, the fluctuation corrections are much more severe. Therefore the saddle-point approximation of the path-integral is not reliable for $d < 4$ and in fact $d = 4$ is the upper critical dimension.