Critical Phenomena - Exercise Set 5

12) XY Model

The XY model is used to describe several condensed matter systems, and consists of two-components spins $\mathbf{S} = (S_x, S_y)$ with $\mathbf{S}^2 = 1$ placed on the sites of a d-dimensional periodic hypercubic lattice. The Hamiltonian is

$$H = -K \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j \tag{11}$$

with $K > 0$, so that the spins want to be aligned. The notation $\langle ij \rangle$ means that i and j are nearest neighbours.

a) What is the symmetry of this system?

b) Express each spin $\mathbf{S}_i$ in terms of an angle θ_i and write the partition function $Z = \int \prod_i d\mathbf{S}_i \, e^{-\beta H}$ in terms of these angles.

c) At low temperature (large β) the spins vary slowly from site to site. Thus we may expand H quadratically in the angles θ_i. What is the resulting low-temperature Hamiltonian?

d) Use a Fourier transformation of the angle variables to diagonalize the low-temperature Hamiltonian.

e) Using Exercise 7, find an expression for $\langle \mathbf{S}_0 \cdot \mathbf{S}_\mathbf{x} \rangle = \mathrm{Re}\langle \exp\left[i(\theta_\mathbf{x} - \theta_0)\right] \rangle$ in the form of an integral. Use the stationary phase approximation to show that for large $|\mathbf{x}|$, the integral is dominated by small momenta. Hence find the asymptotic behaviour of the correlation function $\langle \mathbf{S}_0 \cdot \mathbf{S}_\mathbf{x} \rangle$, depending on the dimension d.

13) Ising 1D Block spin transformation

In this exercise we will learn how to perform a *block-spin transformation* for the 1D Ising model, which is one of the ways of performing a renormalization group transformation. Consider an infinite chain of spins $s_i = \pm 1$ in one dimension with Hamiltonian

$$H = -J \sum_i s_i s_{i+1} - \frac{B}{2} \sum_i (s_i + s_{i+1}) + C \tag{12}$$

It will become clear later why we introduced the constant term. Define the partition function as

$$Z = \sum_{s_1 = \pm 1} \sum_{s_2 = \pm 1} \cdots e^{-H} \tag{13}$$

where we set $\beta = 1$ for simplicity. The idea is to "integrate out" half the spins in the above formula, thus leaving Z unchanged, but expressed in terms of a new Hamiltonian H' which is defined in terms of the remaining spins only.

a) Separate the spins into even and odd and perform explicitly the sum over each odd spin in eq.(13). Show that after "integrating out" all the odd spins the partition function takes the same form as in eq.(13), where now the sum runs over all the even spins and the new Hamiltonian H' is identical to H but with new parameters J', B' and C'. *Hint: consider one single odd spin s and its even nearest neighbours s_- and s_+. Write explicitly all the terms in the Hamiltonian where s appears and perform the sum over $s = \pm 1$. Then consider all the possible values that $s_- + s_+$ may take.*

b) Remarkably, after this complicated procedure we have obtained the same Hamiltonian only with different parameters J', B' and C', defined in terms of the original J, B and C. We may then imagine repeating this procedure multiple times, thus "flowing" in the space of parameters. In the case that $B = 0$, check that no new B term is generated by the flow. Therefore, in this case, find the fixed points of the block-spin transformation. How does the correlation length behave at these fixed points?

14) O(N) model

Consider N scalars $\vec{\phi} = (\phi_1, \dots, \phi_N)$ with Hamiltonian

$$H = \int d^d x \left[\frac{1}{2}(\nabla \vec{\phi})^2 + \frac{1}{2} t \vec{\phi}^2 + g \vec{\phi}^4 \right] \tag{14}$$

which is symmetric under O(N) transformations of the N fields. One can compute the renormalization group beta functions in $d = 4 - \epsilon$ dimensions, which are given by:

$$\frac{dt}{ds} = 2t + \frac{N+2}{2\pi^2} \frac{\Lambda^4}{\Lambda^2 + t} \widetilde{g} \tag{15}$$

$$\frac{d\widetilde{g}}{ds} = \epsilon \widetilde{g} - \frac{N+8}{2\pi^2} \frac{\Lambda^4}{(\Lambda^2 + t)^2} \widetilde{g}^2 \tag{16}$$

where Λ is the cutoff, $\widetilde{g} = \Lambda^{-\epsilon} g$, and the RG transformation is such that $x' = e^s x$.

a) Find the fixed points of the RG flow and show that there is a non-trivial fixed point. This is called the *Wilson-Fisher fixed point*.

b) Linearize the beta functions and study the relevant and irrelevant directions at the Wilson-Fisher fixed point.

c) Compute the critical exponent ν at the Wilson-Fisher fixed point. Assuming that $\eta \sim \epsilon^2$, compute also $\alpha, \beta, \gamma, \delta$ to leading order in ϵ.

d) From the first beta function, we see that t is of order ϵ along the flow from the Gaussian to the Wilson-Fisher fixed point. Under this assumption, find the beta function for $\widetilde{g}$ to leading order in ϵ and solve it.