Critical Phenomena - Solutions to Exercise Set 1

1) Ising 1D Exact Solution

a) The partition function may be written as

$$Z = \sum_{s_1=\pm 1} \cdots \sum_{s_N=\pm 1} \prod_{i=1}^{N} e^{\beta J s_i s_{i+1} + \beta \frac{B}{2}(s_i + s_{i+1})} \tag{1}$$

We give a name to the term in the product,

$$T(s, s') = e^{\beta J s s' + \beta \frac{B}{2}(s+s')} \tag{2}$$

So that we find,

$$Z = \sum_{s_1=\pm 1} \cdots \sum_{s_N=\pm 1} T(s_1, s_2) T(s_2, s_3) \cdots T(s_N, s_1) \tag{3}$$

Now define a matrix T such that $T(s, s')$ are its matrix elements, i.e.

$$T = \begin{pmatrix} T(-1,-1) & T(-1,1) \\ T(1,-1) & T(1,1) \end{pmatrix} = \begin{pmatrix} e^{\beta(J+B)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-B)} \end{pmatrix} \tag{4}$$

Then we recognize that eq.(3) for Z is just the formula for the product of N times the matrix T, plus an overall trace, i.e.

$$(AB)(i,j) = \sum_k A(i,k) B(k,j) \qquad\qquad \mathrm{tr}(A) = \sum_i A(i,i) \tag{5}$$

Then we see that indeed $Z = \mathrm{tr}(T^N)$ as required.

b) It's elementary to compute the eigenvalues of T, which are given by

$$\lambda_\pm = e^{\beta J} \cosh \beta B \pm \sqrt{e^{2\beta J} \sinh^2 \beta B + e^{-2\beta J}} \tag{6}$$

so that we may also compute Z explicitly,

$$Z = \lambda_+^N + \lambda_-^N \tag{7}$$

c) The magnetization can be computed by differentiating the partition function with respect to B,

$$m = \frac{1}{N} \langle \sum_i s_i \rangle \equiv \frac{1}{N} \frac{1}{Z} \sum_{\{s\}} \left(\sum_i s_i \right) e^{-\beta H} = \frac{1}{N \beta Z} \frac{\partial}{\partial B} Z = \frac{1}{\beta} \frac{\lambda_+^{N-1} \frac{\partial \lambda_+}{\partial B} + \lambda_-^{N-1} \frac{\partial \lambda_-}{\partial B}}{\lambda_+^N + \lambda_-^N} \tag{8}$$

We can check that $\lambda_+ > \lambda_-$, so in the thermodynamic limit $Z \approx \lambda_+^N$. We can perform the limit explicitly in the magnetization, and we obtain

$$\lim_{N \to \infty} m = \frac{1}{\beta \lambda_+} \frac{\partial \lambda_+}{\partial B} \tag{9}$$

For $B = 0$ we then find that $\lambda_+|_{B=0} = 2\cosh\beta J$ and (computing the derivative explicitly) $\frac{\partial\lambda_+}{\partial B}|_{B=0} = 0$ so that the magnetization is zero in the thermodynamic limit. For $B \neq 0$ it's easy to compute the formula explicitly, and we will find that the magnetization has the same sign as B, which makes sense.

d) The Hamiltonian is translationally symmetric, so $C(i,j)$ does not depend on the values of i and j but only on their difference $i-j$. For the same reason it doesn't matter what spin we look at, so $\langle s_i\rangle = \langle s_j\rangle$. Now we compute $\langle s_i s_j\rangle$, which is given by

$$\langle s_i s_j\rangle = \frac{1}{Z}\sum_{\{s\}} s_i s_j e^{-\beta H} \tag{10}$$

We write again $e^{-\beta H}$ in terms of the transfer matrix, and we find that

$$\langle s_i s_j\rangle = \frac{1}{Z}\sum_{s_1=\pm1}\cdots\sum_{s_N=\pm1} T(s_1,s_2)T(s_2,s_3)\cdots s_i T(s_i,s_{i+1})\cdots s_j T(s_j,s_{j+1})\cdots T(s_N,s_1) \tag{11}$$

where we have chosen to place s_i just before $T(s_i,s_{i+1})$ and similarly for s_j. So the only difference compared to the original case is that at the two points i and j the matrix elements are modified to

$$\tilde{T}(s,s') = sT(s,s') \tag{12}$$

The new matrix $\tilde{T}$ has matrix elements

$$\tilde{T} = \begin{pmatrix} e^{\beta(J+B)} & e^{-\beta J} \\ -e^{-\beta J} & -e^{\beta(J-B)} \end{pmatrix} = \sigma_z T \tag{13}$$

where σ_z is the third Pauli matrix. Therefore now the expression can again be written as a matrix product plus a trace,

$$\langle s_i s_j\rangle = \frac{1}{Z}\operatorname{tr}(T^{i-1}\,\tilde{T}\,T^{j-i-1}\,\tilde{T}\,T^{N-j}) = \frac{1}{Z}\operatorname{tr}(\sigma_z T^{j-i}\,\sigma_z T^{N-(j-i)}) \tag{14}$$

where we also used the cyclic property of the trace. We see explicitly that $\langle s_i s_j\rangle$ depends only on the difference $j-i$. In order to compute this we use quantum-mechanical notation. Let $v_\pm$ be orthonormal eigenvectors of T, i.e. $Tv_\pm = \lambda_\pm v_\pm$. Then we can write the spectral decomposition of T using bra-ket notation,

$$T^n = \lambda_+^n\,|v_+\rangle\,\langle v_+| + \lambda_-^n\,|v_-\rangle\,\langle v_-| \tag{15}$$

So we find

$$\langle s_i s_j\rangle \times Z = \langle v_+|\,\sigma_z T^{j-i}\,\sigma_z T^{N-(j-i)}\,|v_+\rangle + \langle v_-|\,\sigma_z T^{j-i}\,\sigma_z T^{N-(j-i)}\,|v_-\rangle = \tag{16}$$

$$= \lambda_+^N\,\langle v_+|\,\sigma_z\,|v_+\rangle^2 + +\lambda_-^N\,\langle v_-|\,\sigma_z\,|v_-\rangle^2 + \tag{17}$$

$$+\left(\lambda_-^{N-(j-i)}\lambda_+^{j-i} + \lambda_+^{N-(j-i)}\lambda_-^{j-i}\right)\langle v_+|\,\sigma_z\,|v_-\rangle^2 \tag{18}$$

Remembering that $Z = \lambda_+^N + \lambda_-^N$ and taking the thermodynamic limit $N\to\infty$, since $\lambda_+ > \lambda_-$, we find

$$\langle s_i s_j\rangle = \langle v_+|\,\sigma_z\,|v_+\rangle^2 + \left(\frac{\lambda_-}{\lambda_+}\right)^{j-i}\langle v_+|\,\sigma_z\,|v_-\rangle^2 \tag{19}$$

Repeating the same type of calculation for $\langle s_i \rangle$, it is not hard to see that $\langle s_i \rangle = \langle v_+ | \sigma_z | v_+ \rangle$ in the thermodynamic limit. Therefore we find that

$$C(i,j) = \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle = \left(\frac{\lambda_-}{\lambda_+} \right)^{j-i} \langle v_+ | \sigma_z | v_- \rangle^2 \tag{20}$$

In general it is quite annoying to compute $v_\pm$ with the correct normalization. However, for $B = 0$ we have

$$|v_\pm\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm 1 \\ 1 \end{pmatrix} \tag{21}$$

Therefore we recover the result that $\langle s_i \rangle = \langle v_+ | \sigma_z | v_+ \rangle = 0$ for $B = 0$, and moreover we find $\langle v_+ | \sigma_z | v_- \rangle = -1$. Therefore,

$$C(i,j) = \left(\frac{\lambda_-}{\lambda_+} \right)^{j-i} = (\tanh \beta J)^{j-i} \tag{22}$$

as the eigenvalues also simplify for $B = 0$. We note in passing that the quantum-mechanical notation for T is more than a simple analogy: in fact the transfer matrix for a classical system can be used to find the Hamiltonian of a corresponding quantum-mechanical system in one dimension less. This is known as the *quantum-classical correspondence.*

d) From the expression of part c) we find

$$C(i,j) = e^{(j-i) \log \tanh \beta J} \tag{23}$$

Note that from our construction we assumed implicitly that $j > i$. From the exponential piece we therefore see that

$$\xi = -\frac{1}{\log \tanh \beta J} \tag{24}$$

Since there is no power-law, we conclude that the power is zero, therefore $\eta = 1$ in $d = 1$. Now for $\xi \to \infty$, the denominator must go to 0 (note that $\xi > 0$). This happens when $\tanh \beta J \to 1$, which means that $\beta \to \infty$, so the temperature goes to zero. In fact the 1D Ising model has no critical point at finite temperature. This is consistent with what we found in c).

2) Ising 1D Mean Field Theory

a) As noted during the class, it is not strictly true that $(s_i - m)$ is small for the Ising model, but we actually only need this to be true inside the sum over nearest neighbours, which may or may not be the case. The *Ginzburg criterion*, which we'll see later in the course, will tell us whether mean-field theory works or not. In any case, we set

$$s_i s_j = m^2 + m(s_i - m) + m(s_j - m) \tag{25}$$

in the sum. Then the partition function becomes

$$Z = \sum_{\{s\}} \exp\left(\beta J \sum_i s_i s_{i+1} + \beta B \sum_i s_i\right) = \tag{26}$$

$$= \sum_{\{s\}} \exp\left[\beta J \left(m \sum_i s_i + m \sum_i s_{i+1} - N m^2\right) + \beta B \sum_i s_i\right] = \tag{27}$$

$$= \sum_{\{s\}} \exp\left(-N\beta J m^2 + \beta(2mJ + B) \sum_i s_i\right) \tag{28}$$

We see that the mean-field approximation has removed the interactions. Now the spins only feel the presence of their neighbours through an effective magnetic field $B_{\text{eff}} = B + 2mJ$. All the spins are now indepedent, so

$$Z = \exp\left(-N\beta J m^2\right) \sum_{s_1=\pm1} \sum_{s_2=\pm1} \cdots \sum_{s_N=\pm1} \prod_i \exp\left(\beta(2mJ + B)s_i\right) = \tag{29}$$

$$= \exp\left(-N\beta J m^2\right) \prod_i \left(\sum_{s_i=\pm1} \exp\left(\beta(2mJ + B)s_i\right)\right) = \tag{30}$$

$$= \exp\left(-N\beta J m^2\right) \left[e^{\beta(2mJ+B)} + e^{-\beta(2mJ+B)}\right]^N = \tag{31}$$

$$= \left[2\exp\left(-\beta J m^2\right) \cosh \beta(2mJ + B)\right]^N \tag{32}$$

which is the explicit expression we were looking for.

b) Having obtained the partition function explicitly, we can obtain the magnetization by differentiating wrt B,

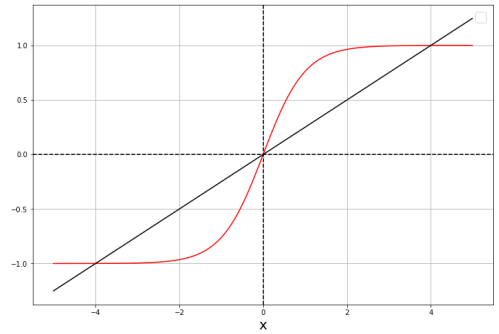
$$m \equiv \frac{1}{N}\langle \sum_i s_i \rangle = \frac{1}{ZN\beta}\frac{\partial Z}{\partial B} \tag{33}$$

Therefore plugging in the mean-field partition function we find,

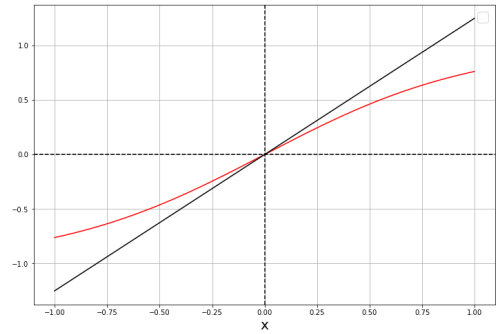
$$m = \tanh\left(2m\beta J + \beta B\right) \tag{34}$$

This is called the self-consistency equation because it ensures that our assumption that the magnetization is m agrees with the magnetization computed under the same assumption. *(Note that there is a mistake in the text of the exercise; the correct statement is that q is the number of nearest neighbours per spin, which means $q = 2$ in our case. Apologies!)*

c) For $B = 0$ the self-consistency equation is given by $m = \tanh(2m\beta J)$. Setting $x = 2m\beta J$, the equation becomes $\frac{x}{2\beta J} = \tanh x$, so it's a matter of finding the intersection between a line and the hyperbolic tangent. Fig.1 shows the geometric problem for two slopes of the line. We see that the two curves intersect either once or three times. This is determined by the relative slopes at $x = 0$. The slope of $\tanh x$ is equal to 1; if the slope of the line is greater than this, then the intersection occurs once; otherwise if the slope of the line is smaller than 1, the intersection occurs three times. Therefore we see that there is a critical value $\beta_c = 1/(2J)$; for $\beta < \beta_c$ there is only one solution, $m = 0$ and the system is disordered. On the other hand for $\beta > \beta_c$, we have three solutions: it turns out



(a) In red $\tanh x$ while in black $0.25x$. The two curves intersect three times.



(b) In red $\tanh x$ while in black $1.25x$. The two curves intersect only once.

Figure 1: Graphical solution of the self-consistency equation.

that $m = 0$ is unstable, so the system will settle at either $\pm m^*$ for some m^*. Thus the system orders and the $\mathbb{Z}_2$ symmetry is spontaneously broken.

In Exercise 1 we solved this system exactly and we found that there was no phase transition. Therefore we see that the mean-field approximation fails completely in this case, because we find a phase transition. It turns out that for the Ising model the mean-field approximation fails in $d = 1$, it is qualitatively correct in $d = 2, 3$ (in the sense that it correctly predicts a phase transition, but gives the wrong critical exponents), while it predicts the correct critical exponents for $d \geq 4$. Thus we say that 2 is the *lower critical dimension* of the Ising model, and 4 is the *upper critical dimension*.

d) For small x, a series expansion gives $\tanh x \approx x$, so in the limit $\beta \to 0$ (high temperature), we find

$$m \approx \frac{\beta B}{(1 - 2\beta J)} \approx \beta B \tag{35}$$

Therefore the magnetization takes the same sign as the magnetic field, as expected.

e) We write $T = T_c(1 + t)$ and we're interested in the behaviour of m for t near zero. Using $\beta_c = 1/(2J)$ and $T = 1/\beta$, we see that

$$m = \tanh\left(\frac{m}{1+t}\right) \tag{36}$$

For $t > 0$, that is $T > T_c$, the magnetization is exactly zero; for small $t < 0$ the system has a small finite magnetization m. Thus the whole argument of the hyperbolic tangent is small, and we may expand it as a series for small values of the parameter. We find,

$$m = \frac{m}{1+t} - \frac{1}{3}\frac{m^3}{(1+t)^3} + \cdots = m - mt - \frac{1}{3}m^3 + \cdots \tag{37}$$

Therefore we see that $m \sim (-t)^{1/2} + \cdots$, so that $\widetilde{\beta} = 1/2$. This is the mean field exponent for the magnetization of the Ising model.