## Critical Phenomena - Solutions to Exercise Set 2

## 3) Ising 2D Self-duality

a) From the exponential definitions of  $\cosh$  and  $\sinh$  it is clear that  $e^x = \cosh x + \sinh x$ . Then the result follows from noting that  $s_i s_j = \pm 1$  and that  $\cosh$  is even while  $\sinh$  is odd. Then it is clear that  $f_0(\beta) = \cosh \beta$  while  $f_1(\beta) = \cosh \beta$ .

b) Suppose that the nearest neighbour pairs are  $\langle 12 \rangle$ ,  $\langle 34 \rangle$ ,  $\langle 13 \rangle$  and so on. Then we can write the product explicitly as

$$\prod_{\langle ij\rangle} \sum_{k=0}^{1} f_k(\beta) (s_i s_j)^k = \left( \sum_{k=0}^{1} f_k(\beta) (s_1 s_2)^k \right) \left( \sum_{k=0}^{1} f_k(\beta) (s_3 s_4)^k \right) \left( \sum_{k=0}^{1} f_k(\beta) (s_1 s_3)^k \right) \cdots$$
(38)

Now technically the ks in each sum are different, so we need to give them a different name. Since there is one term in the product per pair  $\langle ij \rangle$ , we call each k as  $k_{\langle ij \rangle}$ , so that in fact

$$\prod_{\langle ij \rangle} \sum_{k=0}^{1} f_k(\beta)(s_i s_j)^k = \left(\sum_{k_{\langle 12 \rangle}=0}^{1} f_{k_{\langle 12 \rangle}}(\beta)(s_1 s_2)^k\right) \left(\sum_{k_{\langle 34 \rangle}=0}^{1} f_{k_{\langle 34 \rangle}}(\beta)(s_3 s_4)^k\right) \left(\sum_{k_{\langle 13 \rangle}=0}^{1} f_{k_{\langle 13 \rangle}}(\beta)(s_1 s_3)^k\right) \cdots$$
(39)

Now we can group together all the summations and all the products,

$$\prod_{\langle ij\rangle} \sum_{k=0}^{1} f_k(\beta)(s_i s_j)^k = \sum_{k_{\langle 12\rangle}=0}^{1} \sum_{k_{\langle 34\rangle}=0}^{1} \sum_{k_{\langle 13\rangle}=0}^{1} \cdots \left( f_{k_{\langle 12\rangle}}(\beta)(s_1 s_2)^k \right) \left( f_{k_{\langle 34\rangle}}(\beta)(s_3 s_4)^k \right) \left( f_{k_{\langle 13\rangle}}(\beta)(s_1 s_3)^k \right) \cdots$$

$$(40)$$

Now we see that we have one k per nearest neighbour pair  $\langle ij \rangle$ . So a configuration  $\{k\}$  of ks is an assignment of one  $k_{\langle ij \rangle} = 0, 1$  to each nearest neighbour pair. Therefore by definition,

$$\sum_{\{k\}} \equiv \sum_{k_{\langle 12\rangle}=0}^{1} \sum_{k_{\langle 34\rangle}=0}^{1} \sum_{k_{\langle 13\rangle}=0}^{1} \cdots$$
(41)

We're then left with simply the product over all nearest neighbour pairs inside this sum,

$$\prod_{\langle ij\rangle} \sum_{k=0}^{1} f_k(\beta) (s_i s_j)^k = \sum_{\{k\}} \prod_{\langle ij\rangle} f_{k_{\langle ij\rangle}}(\beta) (s_i s_j)^k$$
(42)

which is the formula that we needed. This technique of exchanging summation and product is often very useful and of very general applicability: it doesn't depend on the range of k, nor on the number of factors in the product, nor on the form of the object we're summing over.

c) Now the partition function is therefore given by

$$Z(\beta) = \sum_{\{s\}} \sum_{\{k\}} \prod_{\langle ij \rangle} f_{k_{\langle ij \rangle}}(\beta) (s_i s_j)^{k_{\langle ij \rangle}}$$
(43)

We can break up the last product into two pieces,

$$Z(\beta) = \sum_{\{s\}} \sum_{\{k\}} \left( \prod_{\langle ij \rangle} f_{k_{\langle ij \rangle}}(\beta) \right) \prod_{\langle ij \rangle} (s_i s_j)^{k_{\langle ij \rangle}}$$
(44)

Now we want to turn the last product from a product over links (nearest neighbour pairs) to a product over single sites. To gain some intuition, we note that if again the nearest neighbour pairs are  $\langle 12 \rangle$ ,  $\langle 34 \rangle$ ,  $\langle 13 \rangle$  and so on, then the product is

$$\prod_{\langle ij\rangle} (s_i s_j)^{k_{\langle ij\rangle}} = (s_1 s_2)^{k_{\langle 12\rangle}} (s_3 s_4)^{k_{\langle 34\rangle}} (s_1 s_3)^{k_{\langle 13\rangle}} \cdots$$
(45)

We want to gather together all the terms corresponding to each individual site *i*. We see that each spin  $s_i$  will appear in the product as  $s_i^{k_{\langle ij \rangle}}$  for each of its nearest neighbours *j*. Therefore gathering all these terms together we get  $s_i^{\sum_{\langle ij \rangle} k_{\langle ij \rangle}}$  where the sum runs over all nearest neighbour pairs emanating from *i*. Therefore we find,

$$Z(\beta) = \sum_{\{s\}} \sum_{\{k\}} \left( \prod_{\langle ij \rangle} f_{k_{\langle ij \rangle}}(\beta) \right) \prod_{i} s_{i}^{\sum_{\langle ij \rangle} k_{\langle ij \rangle}}$$
(46)

Now we can move the sum over all spin configurations to the end. Since this is a sum over each individual spin  $s_i$ , overall we find

$$Z(\beta) = \sum_{\{k\}} \left( \prod_{\langle ij \rangle} f_{k_{\langle ij \rangle}}(\beta) \right) \prod_{i} \left( \sum_{s_i = \pm 1} s_i^{\sum_{\langle ij \rangle} k_{\langle ij \rangle}} \right)$$
(47)

which is the formula we were looking for. Now we can compute the sum explicitly,

$$\sum_{s_i=\pm 1} s_i^{\sum_{\langle ij\rangle} k_{\langle ij\rangle}} = 1 + (-1)^{\sum_{\langle ij\rangle} k_{\langle ij\rangle}}$$
(48)

Therefore it equals zero if  $\sum_{\langle ij \rangle} k_{\langle ij \rangle}$  is odd, and it equals 2 otherwise. Now looking at the form of the partition function, we see that it is now a sum over all configurations of k. However, because of this last product, many of these configurations do not contribute to the partition function; the ones that contribute are those for which  $\sum_{\langle ij \rangle} k_{\langle ij \rangle}$  is even for *all* sites *i*. Therefore, in some sense, the *k*s are not the *correct* variables, because only some of the *k* configurations contribute to the partition function. It is then natural to try to express *k* in terms of new variables such that the evenness of the sum is always true.

d) The dual lattice is drawn in Fig.2a. The sites of the original lattice are the intersections of the solid lines, while the sites of the dual lattice are the intersections of the dotted lines. So we see that each site of the original lattice is enclosed by a unique plaquette (smallest square) in the dual lattice and viceversa. We also see that each link (a connection between two neighbouring sites) crosses a unique link in the dual lattice (i.e. the two together form a cross).

As we discussed in class, in this case the introduction of the dual lattice may seem to be insufficiently motivated. After all, it's just another square lattice! However, it turns out



(a) A two dimensional square lattice (solid (b) A site in the original lattice with its four line) and its dual lattice (dotted line). neighbours and the dual links.

Figure 2: The original lattice and its dual

that this is a good idea in the long run for two reasons: the first one is that the original and dual lattice have a natural interpretation in terms of a mathematical construct called *lattice differential forms* which makes it much easier to construct dualities. The second reason is that it generalizes to more complicated lattices than the square lattice; for example one can consider the Ising model on the honeycomb lattice, then the dual theory is again an Ising model which lives on the dual lattice, which is now a triangular lattice. Without the concept of a dual lattice one might have tried to find a duality to another honeycomb lattice, which wouldn't have worked. So this concept generalizes nicely.

e) We have seen in d) that each link in the original lattice identifies a link in the dual lattice. Each link is nothing but a nearest neighbour pair. The relation is well defined because  $\sigma_{\tilde{i}} \sigma_{\tilde{j}} = \pm 1$  so  $\frac{1}{2}(1 - \sigma_{\tilde{i}} \sigma_{\tilde{j}}) \in \{0, 1\}$  as required. Looking at Fig.2b, we see that the nearest neighbours of a site *i* in the original lattice form a "star" or a "cross" made of four links. Then the dual links of the four links form a plaquette in the dual lattice; at the four vertices of the plaquette we have four dual variables which we call  $\sigma_{\tilde{i}}$  for  $\tilde{i} = 1, 2, 3, 4$ . Then we see that

$$\sum_{\langle ij\rangle} k_{\langle ij\rangle} = 2 - \frac{1}{2} \left( \sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_4 + \sigma_4 \sigma_1 \right)$$
(49)

Then the sum inside the bracket is always a multiple of four. This is perhaps surprising because naively we might think it is just a multiple of two. However, the four terms are not independent: we can pick  $x = \sigma_1 \sigma_2$ ,  $y = \sigma_2 \sigma_3$ ,  $z = \sigma_3 \sigma_4$  independently from  $\pm 1$  but then  $xyz = \sigma_1 \sigma_2 \sigma_2 \sigma_3 \sigma_3 \sigma_4 = \sigma_1 \sigma_4$  is the last term. Then the quantity in brackets becomes

$$x + y + z + xyz = x + y + z(1 + xy)$$
(50)

If x and y have opposite signs then x + y = 0 and 1 + xy = 0, so we get zero, which is a multiple of four. On the other hand if they have the same sign then x = y so overall we get 2x + 2z, which is either zero or four. So overall  $(\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_4 + \sigma_4 \sigma_1)$  is always a multiple of four, meaning that the sum over the ks is always even.

What we did here is we started from some variables (the ks) which were not very good because only a subset of the configurations of ks contributed to the partition function. Then we expressed k in terms of new variables (the  $\sigma s$ ) such that now all the ks in this form contribute. In this case we postulated the relation between the ks and the  $\sigma s$  but there is in fact a mathematical theory (which we won't discuss) which allows us to *derive*  this relation. In fact it can be shown that if we want  $\sum_{\langle ij \rangle} k_{\langle ij \rangle}$  to be even for *every site* i, then it must be the case that  $k_{\langle ij \rangle} = \frac{1}{2}(1 \pm \sigma_{\tilde{i}} \sigma_{\tilde{j}})$  for some new variables  $\sigma = \pm 1$ .

f) Expressing k terms of the new variables  $\sigma$  we have

$$\sum_{s_i=\pm 1} s_i^{\sum_{\langle ij \rangle} k_{\langle ij \rangle}} = 2 \tag{51}$$

because the sum is always even. So we get a factor of 2 for each site and there's N of them. So we find

$$Z(\beta) = 2^{N} \frac{1}{2} \sum_{\{\sigma\}} \left( \prod_{\langle \tilde{i}\tilde{j} \rangle} f_{\frac{1}{2} \left(1 - \sigma_{\tilde{i}} \sigma_{\tilde{j}}\right)}(\beta) \right)$$
(52)

where the product now runs over nearest neighbour pairs in the dual lattice (which as we have seen are in one-to-one correspondence with nearest neighbour pairs in the original lattice). We also replaced the sum over ks with the sum over  $\sigma$ s, but since if we replace  $\sigma$  with  $-\sigma$  the ks are unchanged, by doing so we're overcounting by a factor of 2, which we therefore divide by.

g) Since k = 0, 1 the result follows easily in each of the two cases. Then plugging it in the partition function, we find

$$Z(\beta) = 2^{N} \frac{1}{2} \sum_{\{\sigma\}} \prod_{\langle ij \rangle} \cosh(\beta) \exp\left[\frac{1}{2} \left(1 - \sigma_{\tilde{i}} \sigma_{\tilde{j}}\right) \log \tanh(\beta)\right] =$$

$$= 2^{N} \frac{1}{2} \sum_{\{\sigma\}} \left(\prod_{\langle ij \rangle} \cosh(\beta) \exp\left[\frac{1}{2} \log \tanh(\beta)\right]\right) \exp\left[-\frac{1}{2} \log \tanh(\beta) \sum_{\langle ij \rangle} \sigma_{\tilde{i}} \sigma_{\tilde{j}}\right] =$$
(53)

$$= \frac{(\sinh(2\beta))^{N}}{2} \sum_{\{\sigma\}} \exp\left[-\frac{1}{2}\log\tanh(\beta)\sum_{\langle \tilde{i}\tilde{j}\rangle}\sigma_{\tilde{i}}\sigma_{\tilde{j}}\right]$$
(55)

where in going to the last line we noted that the terms inside the product are now constant. On a lattice with N spins, we have 2N nearest neighbour pairs, so 2N terms in the product and the rest follows from some algebra. The important thing to note is that on the last line we have again an expression which is identical to the partition function of the 2D Ising model at inverse temperature  $\beta^* = -\frac{1}{2} \log \tanh(\beta)$ . Therefore,

$$Z(\beta) = \frac{\left(\sinh\left(2\beta\right)\right)^N}{2} Z(\beta^*) \tag{56}$$

We summarise this fact by saying that the 2D Ising model is *self-dual*: after performing the duality transformation, we have obtained the same partition function at a different temperature. As we will see shortly, the prefactor does not play any role and can generally be ignored. It's not difficult to see that  $\beta^*$  is a decreasing function of  $\beta$ ; and moreover  $\beta^* \to 0$  as  $\beta \to \infty$ , while  $\beta^* \to \infty$  as  $\beta \to 0$ . This happens generically: dualities swap high-temperature with low-temperature. h) The free energy per spin is defined in the thermodynamic limit as

$$f(\beta) = \lim_{N \to \infty} -\frac{1}{N\beta} \log Z(\beta)$$
(57)

Therefore taking the logs on both sides of eq.(56) and performing the appropriate limit, we find that

$$f(\beta) = -\frac{1}{\beta} \log \sinh \left(2\beta\right) + f(\beta^*) \tag{58}$$

Phase transitions occur at a point of non-analiticity of  $f(\beta)$ . Now the term in the middle,  $-\frac{1}{\beta} \log \sinh(2\beta)$ , is analytic for any finite  $\beta$ , and therefore does not matter for phase transitions.

Now suppose that there is a phase transition at  $\beta = \beta_c$ . This means that  $f(\beta)$  is nonanalytic at  $\beta_c$ . But from the above relation eq.(58), this also means that  $f(\beta)$  is nonanalytic also at  $\beta_c^* = -\frac{1}{2} \log \tanh(\beta_c)$ . So if there is a phase transition at  $\beta_c$ , there must also be a phase transition at  $\beta_c^*$ . Therefore we have a relation between the points at which phase transitions occur. Assuming that there is only one phase transition, this means that we must have  $\beta_c = \beta_c^*$ . This equation can be solved exactly as follows. We have:

$$\beta_c = -\frac{1}{2}\log \tanh(\beta_c) \quad \rightarrow \quad e^{-2\beta_c} = \tanh(\beta_c)$$
(59)

This is good because we know that also tanh can be expressed in terms of exponentials. But it would be even better to express it in terms of  $e^{2\beta_c}$ , and in fact,

$$\tanh \beta_c \equiv \frac{\sinh \beta_c}{\cosh \beta_c} = \frac{\sinh \beta_c \cosh \beta_c}{\cosh^2 \beta_c} = \frac{e^{2\beta_c} - e^{-2\beta_c}}{e^{2\beta_c} + e^{-2\beta_c} + 2} \tag{60}$$

Now substituting  $x = e^{2\beta_c}$  and simplifying, this becomes an algebraic equation,  $x^3 - x^2 - 3x - 1 = 0$ . This is easy to solve because it has a root x = -1, while the other two roots are  $x = 1 \pm \sqrt{2}$ . Only the positive root is acceptable, so in the end we find

$$\beta_c = \frac{1}{2}\log\left(1 + \sqrt{2}\right) \tag{61}$$

which is the transition temperature.

## 4) Ginzburg-Landau Theory

a) Considering the terms up to  $m^4$  we see that if c(T) < 0, then  $f(m) \to -\infty$  as  $m \to \pm \infty$ . Thus the system is unstable. On the other hand as long as c(T) > 0 the system has a global minimum and is therefore stable.

b) To find the minima of the free energy we differentiate f to find,

$$\frac{\partial f}{\partial m} = b(T)m + \frac{1}{3}c(T)m^3 = 0 \qquad \rightarrow \qquad m = 0, \quad m = \pm \sqrt{\frac{-b(T)}{c(T)}} \tag{62}$$

Therefore we have two cases: if b(T) > 0 then only m = 0 is a solution and it's not hard to see that it is a global minimum. On the other hand if b(T) < 0 then all three solutions



(a) Plot of the free energy for b(T) > 0. (b) Plot of the free energy for b(T) < 0.

Figure 3: Sketch of the free energy  $f(m) = a(T) + \frac{1}{2}b(T)m^2 + \frac{1}{4}c(T)m^4$ .

are allowed. We find that m = 0 is now a maximum, while  $m = \pm \sqrt{\frac{-b(T)}{c(T)}}$  are minima. The  $\mathbb{Z}_2$  symmetry is broken by a choice of one of the two minima. The free energy is plotted in Fig.3. The fact that m = 0 is a local maximum and therefore an unstable equilibrium confirms what we said in the solution to Exercise 2. So in the end if we have a phase transition at  $T = T_c$  then b(T) must change sign at  $T_c$ .

c) To its lowest order,  $b(T) = b(T - T_c)$  with b > 0 so at high temperature the system is disordered, while it is ordered at low temperature as we expect. The minimum of the free energy is therefore m = 0 for  $T > T_c$  and  $m = \pm \sqrt{\frac{b}{c}(T_c - T)}$  for  $T < T_c$ .

d) From the solution in part c) and the definition in Exercise 2, we find  $\tilde{\beta} = 1/2$  which is the same answer we found in Exercise 2. This is the mean-field exponent. In fact when we wrote down the free energy in this exercise, we ignored the fluctuations in m, which is the same as the mean-field approximation of Exercise 2. Note that we assumed nothing about the dimensionality of the system in this exercise.

## 5) Tricritical point

a) First of all, we must have c > 0 in order for the system to be stable. Then the critical points are given by

$$\frac{\partial f}{\partial m} = 2m(a(T) + 2bm^2 + 3cm^4) = 0 \qquad \to \qquad m = 0, \qquad m^2 = \frac{-b \pm \sqrt{b^2 - 3ca(T)}}{3c}$$
(63)

The second solution is admissible only if  $b^2 - 3ca(T) > 0$  and  $m^2 > 0$ . Then we have to check several cases.

We sketch the phase diagram first for b > 0. The solution will depend on a(T). If a(T) > 0 we can check that only m = 0 is a solution, and it is a global minimum. On the other hand, if a(T) < 0 we can check that we have three critical points at m = 0 and  $m = \pm \sqrt{\frac{-b + \sqrt{b^2 - 3ca(T)}}{3c}}$ . To determine the global minimum we plug the values back into f (for the non-trivial points this can be simplified by using their defining equation) and we see that now the two non-trivial critical points are global minima. Therefore for b > 0

we have a phase transition as a(T) changes sign, and we also see that the magnetization continuously goes from non-zero to zero as we increase a(T) from positive to negative. Hence the transition is second-order.

Now consider the case b < 0. In that case if  $a(T) > b^2/(3c)$  then only m = 0 is a solution, and it is a global minimum. If, on the other hand,  $0 < a(T) < b^2/(3c)$  then five critical points emerge at m = 0 and  $m = \pm \sqrt{\frac{-b \pm \sqrt{b^2 - 3ca(T)}}{3c}}$ . The solutions with - inside the square root are local maxima, while the ones with + are local minima. If we decrease a(T) further so that a(T) < 0 then we find three critical points at m = 0 and  $m = \pm \sqrt{\frac{-b \pm \sqrt{b^2 - 3ca(T)}}{3c}}$ . We then need to determine the global minimum, and we find that for  $a(T) > b^2/(4c)$  the global minimum is  $m = \pm \sqrt{\frac{-b \pm \sqrt{b^2 - 3ca(T)}}{3c}}$ . This time the transition occurs as a(T) crosses  $b^2/(4c)$ . Hence m is discontinuous at the transition, which is therefore first order.

b) For b = 0 then the situation simplifies substantially and we have critical points at m = 0 and  $m = \left(-\frac{a(T)}{3c}\right)^{1/4}$ . Therefore the phase transition occurs at a(T) = 0. Expanding  $a(T) = a(T - T_c)$  like in Exercise 4, we see that  $m = \left(\frac{a}{3c}(T_c - T)\right)^{1/4}$  so that the magnetization critical exponent  $\beta = 1/4$ . Substituting into the free energy we find,

$$f = -\frac{2}{3} \left(\frac{a^3}{3c}\right)^{1/2} (T_c - T)^{3/2}$$
(64)

Then the heat capacity is

$$C = -T\frac{\partial^2 f}{\partial T^2} = \frac{1}{2} \left(\frac{a^3}{3c}\right)^{1/2} \frac{T}{\sqrt{T_c - T}}$$
(65)

Therefore we see that the critical exponent for the heat capacity is  $\alpha = 1/2$ . For other critical exponents, we add a magnetic field, so that

$$f(m,T) = a(T)m^{2} + c m^{6} - Bm$$
(66)

For the critical exponent  $\delta$  we go to the phase transition so a(T) = 0 and we see how the equilibrium magnetization varies. We find

$$\frac{\partial f}{\partial m} = 6c \, m^5 - B \qquad \rightarrow \qquad B = 6c \, m^5 \tag{67}$$

so that  $\delta = 5$ . Then finally restoring a(T) we find

$$\frac{\partial f}{\partial m} = 2\alpha(T)m + 6c\,m^5 - B \qquad \rightarrow \qquad B = 2a(T)m + 6c\,m^5 \tag{68}$$

Then we probe how m varies with B, finding

$$\chi = \frac{\partial m}{\partial B} = \frac{1}{2a(T) + 30c \, m^4} \tag{69}$$

Therefore near the phase transition  $\chi \sim 1/(T_c - T)$  so  $\gamma = 1$ .