Critical Phenomena - Solutions to Exercise Set 3

6) Superfluid order parameter

a) For a space-independent order-parameter the free energy reduces to

$$f = a(T)\,|\psi|^2 + b\,|\psi|^4 \tag{70}$$

where $f = F/V$ is the free energy per unit volume. Remembering that ψ and ψ^* are independent variables in complex analysis, we see that

$$\frac{\partial f}{\partial \psi} = \psi^* \left(a(T) + 2b\,|\psi|^2\right) \tag{71}$$

So we have minima at $\psi = 0$ and $|\psi| = \sqrt{\frac{-a(T)}{2b}}$. Therefore we have a phase transition as $a(T)$ changes sign. This is associated with spontaneous symmetry breaking; in fact the free energy has a symmetry $\psi \to e^{i\theta}\psi$ (this is a U(1) symmetry as $e^{i\theta} \in$ U(1)). The vacuum $\psi = 0$ is symmetric under this; however on the other side the vacuum is $\psi = e^{i\varphi}\sqrt{\frac{-a(T)}{2b}}$ for an arbitrary φ, which is not invariant under the symmetry.

Again expanding $a(T) = a(T - T_c)$ for some constant $a > 0$, we see that $|\psi| \sim \sqrt{T - T_c}$ so the critical exponent $\beta = 1/2$. Substituting back into the free energy, we see that $f = -\frac{a(T)^2}{4b}$ and therefore $C \sim -T\frac{\partial^2 f}{\partial T^2} \sim$ constant. Therefore $\alpha = 0$. To compute the other two critical exponent we add a linear term $-B\,|\psi|$ to the free energy. At the phase transition $a(T) = 0$ so $f = -B\,|\psi| + b\,|\psi|^4$. Minimising wrt $|\psi|$ we find $|\psi| \sim B^{1/3}$ so $\delta = 3$. Finally, keeping only the smallest terms, $f = -B\,|\psi| + a(T)\,|\psi|^2$ so $m \sim \frac{B}{T-T_c}$. Therefore $\chi = \frac{\partial m}{\partial B}|_T \sim \frac{1}{T-T_c}$. So $\gamma = 1$. These are the same critical exponents as for the mean field Ising model.

The magnetic term In fact $B\,|\psi|$ is not a good magnetic term, because $|\psi| > 0$ while we would expect a magnetic field to force the order parameter to align in the same direction. To achieve this, pick B complex and write the magnetic term as $-(B^*\psi + B\psi^*) \propto -\cos\varphi_B - \varphi_\psi$ where φ_X is the complex phase of variable. Now then the phase of ψ wants to align with B and this is a better magnetic field term. In any case critical exponents are robust, so both these terms give the same answer.

b) We expand the field in momentum space,

$$\psi(x) = \int \frac{dk}{2\pi} e^{ikx}\widetilde{\psi}(k) \tag{72}$$

Then we can substitute into the free energy and we obtain,

$$\begin{aligned} F = &\int \frac{dk}{2\pi}\left(a(T) - \gamma k^2\right)\left|\widetilde{\psi}(k)\right|^2 + \\ &+ \int \frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\frac{dk_3}{2\pi}\frac{dk_4}{2\pi}\, 2\pi\delta(k_1 + k_2 - k_3 - k_4)\,(b + \mu k_1 k_2 k_3 k_4)\widetilde{\psi}(k_1)\widetilde{\psi}(k_2)\widetilde{\psi}(k_3)^*\widetilde{\psi}(k_4)^* \end{aligned} \tag{73}$$

where we used the definition of the delta function, $\delta(k) = \int \frac{dx}{2\pi} e^{ikx}$.

c) If only one Fourier mode contributes, this means that we can write the ansatz $\psi(x) = Ae^{ikx}$. Substituting into the free energy, we find for $f = F/V$

$$f = (a(T) - \gamma k^2) |A|^2 + (b + \mu k^4) |A|^4 \tag{74}$$

To minimize f we differentiate wrt k and $|A|$ and we see that the minimum is achieved

$$\frac{\partial f}{\partial k} = -2\gamma k |A|^2 + 4\mu k^3 |A|^4 = 0 \qquad \to \qquad |A| = 0, \quad k = 0, \quad k^2 = \frac{\gamma}{2\mu |A|^2} \tag{75}$$

and

$$\frac{\partial f}{\partial |A|} = 2|A| \left[(a(T) - \gamma k^2) + 2|A|^2 (b + \mu k^4)\right] = 0 \tag{76}$$

So we find three solutions. The first one is $|A| = 0$, the second one is $k = 0$, $|A|^2 = -\frac{a(T)}{2b}$, while the third solution is $k = -\frac{\gamma b}{\mu a(T)}$, $|A|^2 = -\frac{a(T)}{2b}$. The latter two exist only for $a(T) < 0$. Therefore we can check that the global minimum is $\psi = 0$ for $a(T) > 0$ and the third solution for $a(T) < 0$. We therefore have a phase transition at $a(T) = 0$. The choice of ansatz for ψ breaks the U(1) symmetry explicitly, but we still have a symmetry $k \to -k$ because the solutions only depend on k^2. This $\mathbb{Z}_2$ symmetry is spontaneously broken.

Bonus In fact I confess that I made a mistake in writing down the exercise and that the last term should have been $\left|\frac{d^2\psi}{dx^2}\right|^2$ instead of $\left|\frac{d\psi}{dx}\right|^4$. So we work out this case as well.

In the momentum space free energy this term would simply give rise to a term $\mu k^4 \left|\widetilde{\psi}(k)\right|^2$. As for part c) the situation is not so different. We have

$$f = (a(T) - \gamma k^2 + \mu k^4) |A|^2 + b |A|^4 \tag{77}$$

So that we find

$$\frac{\partial f}{\partial k} = (-2\gamma + 4\mu k^2) k |A|^2 \qquad\qquad \frac{\partial f}{\partial |A|} = 2|A| \left[(a(T) - \gamma k^2 + \mu k^4) + 2b|A|^2\right] \tag{78}$$

So we find again three critical points. One of them is $|A| = 0$ while the other two are $k^2 = \frac{\gamma}{2\mu}$, $|A| = \frac{-2a(T) + \gamma^2/(2\mu)}{4b}$ (for $a(T) < \gamma^2/(4\mu)$) and $k = 0$, $|A|^2 = -\frac{a(T)}{2b}$ for $a(T) < 0$. However we can check that the global minimum is $\psi = 0$ for $a(T) > \gamma^2/(4\mu)$ and the second solution (with $k \neq 0$) for $a(T) < \gamma^2/(4\mu)$. So we have a single phase transition at $a(T) = \gamma^2/(4\mu)$.

7) Wick's identity

a) The condition $\langle 1 \rangle = 1$ means that

$$Z = \int_{-\infty}^{+\infty} d^N\phi \, \exp\left[-\frac{1}{2}\phi \cdot G^{-1} \cdot \phi\right] \tag{79}$$

In order to compute this note that $\phi \cdot G^{-1} \cdot \phi \equiv \sum_{ij} \phi_i (G^{-1})_{ij} \phi_j$ is symmetric under the exchange of ϕ_i and ϕ_j so we may assume without loss of generality that G^{-1} is symmetric. Therefore it can be diagonalized, $G^{-1} = O^T D O$ where D is diagonal and O orthogonal. Now we make the change of variables $\tilde{\phi} = O\phi$. The Jacobian of this transformation is $|\det O| = 1$ because O is orthogonal, so we find,

$$Z = \int_{-\infty}^{+\infty} d^N\tilde{\phi} \, \exp\left[-\frac{1}{2}\tilde{\phi} \cdot D \cdot \tilde{\phi}\right] \tag{80}$$

The diagonal entries of D are simply the eigenvalues λ_i of G^{-1}, so that the integral factorizes,

$$Z = \prod_i \int_{-\infty}^{+\infty} d\tilde{\phi}_i \, \exp\left[-\frac{1}{2}\lambda_i \tilde{\phi}_i^2\right] = \prod_i \sqrt{\frac{2\pi}{\lambda_i}} = (2\pi)^{N/2} (\det G)^{1/2} \tag{81}$$

where now no summation is implied, and we computed the Gaussian integrals explicitly. The final equality follows because $\prod_i \lambda_i = \det(G^{-1}) = (\det G)^{-1}$.

b) As suggested by the hint, we add a term $J \cdot \phi \equiv \sum_i J_i \phi_i$ and define,

$$Z[J] = \int_{-\infty}^{+\infty} d^N\phi \, \exp\left[-\frac{1}{2}\phi \cdot G^{-1} \cdot \phi + J \cdot \phi\right] \tag{82}$$

Therefore we find

$$\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j} Z[J] = \frac{1}{Z}\int_{-\infty}^{+\infty} d^N\phi \; \phi_i \phi_j \, \exp\left[-\frac{1}{2}\phi \cdot G^{-1} \cdot \phi + J \cdot \phi\right] \tag{83}$$

So that therefore,

$$\langle \phi_i \phi_j \rangle = \frac{1}{Z}\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j} Z[J]\bigg|_{J=0} \tag{84}$$

Now we compute $Z[J]$ explicitly. To do so, we complete the square,

$$-\frac{1}{2}\phi^T G^{-1}\phi + J^T\phi = -\frac{1}{2}(\phi - GJ)^T G^{-1}(\phi - GJ) + \frac{1}{2}J^T G J \tag{85}$$

where we used matrix-vector notation. Substituting into $Z[J]$ and changing variables $\phi \to \phi - GJ$ we see that $Z[J]$ reduces to Z times a prefactor,

$$Z[J] = \exp\left(\frac{1}{2}J^T G J\right) Z \tag{86}$$

Now we can compute the derivatives explicitly and we find,

$$\langle \phi_i \phi_j \rangle = \frac{1}{Z}\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j} Z[J]\bigg|_{J=0} = G_{ij} \tag{87}$$

c) We've basically done all the work already in part b). Using the notation of the previous section we have

$$\left\langle \exp\left(\sum_i A_i \phi_i\right)\right\rangle = \frac{1}{Z} \times Z[A] = \exp\left(\frac{1}{2}A^T G A\right) \tag{88}$$

where we used the explicit form of $Z[A]$. The result then follows because we have shown that $\langle \phi_i \phi_j \rangle = G_{ij}$.

8) Asymptotic behaviour of the Green's function

a) Since both k^2 and the integral measure are rotationally invariant, for each fixed $\mathbf{x}$ we are free to rotate the coordinate system of the $\mathbf{k}$ so that $\mathbf{x} = (0, 0, r)$ with $r = |\mathbf{x}| > 0$. Then we see that the integral only depends on r.

The integral identity is valid for all $A > 0$ such that the integral is convergent. Since $k^2 + 1/\xi^2 > 0$ we can apply it with $A = k^2 + 1/\xi^2$ and find,

$$G(\mathbf{x}) = \int_0^\infty dt \int \frac{d^d k}{(2\pi)^d} e^{-i\mathbf{k}\cdot\mathbf{x}} e^{-t(k^2+1/\xi^2)} = \int_0^\infty dt \, \frac{1}{(4\pi t)^{d/2}} e^{-\frac{r^2}{4t} - t/\xi^2} \tag{89}$$

where the inner integral is nothing but the Fourier transform of a d-dimensional Gaussian, which can be computed by completing the square:

$$\int_{-\infty}^{+\infty} dk \, e^{-ikx} e^{-bk^2} = \sqrt{\frac{\pi}{b}} e^{-\frac{x^2}{4b}} \tag{90}$$

Therefore $G(r) = \int_0^\infty dt \, \exp\left(-S(t)\right)$ where

$$S(t) = \frac{d}{2} \log\left(4\pi t\right) + \frac{r^2}{4t} + t/\xi^2 \tag{91}$$

b) Differentiating wrt t, we find

$$S'(t) = \frac{d}{2t} - \frac{r^2}{4t^2} + 1/\xi^2 = 0 \qquad \rightarrow \qquad t_* = \frac{r^2}{d \pm \sqrt{d^2 + 4r^2/\xi^2}} \tag{92}$$

Since the range of integration is $t > 0$ only the $+$ solution is acceptable. Then we approximate,

$$\int_0^\infty dt \, e^{-S(t)} \approx \int_0^\infty dt \, e^{-S(t_*) - S''(t_*)t^2/2} = \sqrt{\frac{\pi}{2S''(t_*)}} e^{-S(t_*)} \tag{93}$$

Note that we also had to approximate the integration interval, otherwise we would have been unable to perform the integration explicitly.

Now if $r \ll \xi$ then $t_* \approx \frac{r^2}{2d}$ and we find $S''(t_*) = \frac{2d^3}{r^4}$, while

$$S(t_*) = \frac{d}{2} + \frac{r^2}{2d\xi^2} + \frac{d}{2} \log \frac{\pi r^2}{d} \approx \text{const} + d \log r \tag{94}$$

Therefore we find

$$G(r) \sim \frac{1}{r^{d-2}} \tag{95}$$

On the other hand if $r \gg \xi$ we find $t_* \approx \frac{2}{r\xi}$ and $S''(t_*) = \frac{4}{\xi^3 r}$. Moreover $S(t_*) = \frac{r}{\xi} + \frac{d}{2} \log\left(\xi r\right)$ so that overall

$$G(r) \sim \frac{e^{-r/\xi}}{r^{(d-1)/2}} \tag{96}$$

9) More asymptotic behaviour

a) This is just spherical coordinates in d dimensions, we can look up the integration measure. The angular integration trivially gives the area of the d-sphere, so that

$$I_d = \frac{S_d}{(2\pi)^d} \int_0^\infty dk \frac{k^{d-1}}{(k^2 + 1/\xi^2)^2} \tag{97}$$

The denominator is always positive and bounded below, so it gives no issues. Therefore the only problems in this integral could arise as $k \to \infty$.

b) The divergences arise as $k \to \infty$, so for large enough k we can ignore the $1/\xi^2$ factor in the denominator, and therefore,

$$I_d \sim \int_0^\infty dk\, k^{d-5} \sim k^{d-4}|_0^\infty = \infty \tag{98}$$

for $d > 4$. So introducing a momentum cutoff Λ we find

$$I_d \sim \frac{S_d}{(2\pi)^d} \int_0^\Lambda dk\, k^{d-5} \sim \frac{S_d}{(2\pi)^d} \Lambda^{d-4} \tag{99}$$

c) The same calculation for $d < 4$ shows that for large k the integral stays finite. In fact we can extract the dimensional dependence of I_d by performing a change of variables. Setting $u = \xi k$, we find

$$I_d = \frac{S_d}{(2\pi)^d} \xi^{4-d} \int_0^\infty du \frac{u^{d-1}}{(u^2 + 1)^2} \tag{100}$$

where the u integral is a finite number for $d < 4$.