3 Dirac Algebra

3.1 Levi-Civita symbol

Vectors or tensors under Lorentz transformations carry Greek indices μ, ν, τ, ... In $d = 4$ space-time dimensions, the indices take the values $0, 1, 2, 3$. The coordinates of a space-time point are written as x^μ with $x^0 = t$ (we set $c = 1$) and $\vec{x} = \sum_i x^i \vec{e}_i$. We use Latin indices to denote the space components x^i with $i = 1, 2, 3$. The four-dimensional metric in Minkowski space-time is written in the form

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{1}$$

Often also the notation $\eta_{\mu\nu}$ is used in flat Minkowski space-time. The inverse metric $g^{\mu\nu}$ is numerically equal to $g_{\mu\nu}$ and satisfies, by definition,

$$g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho \equiv \delta^\mu{}_\rho \, , \qquad g^{\mu\nu} g_{\mu\nu} = \delta^\mu_\mu = d = 4 \, .$$

We employ the Einstein convention and sum over repeated indices. It is good to note that in general $M^\mu{}_\nu \neq M_\nu{}^\mu$, however δ is symmetric so the simple notation δ^μ_ν can be used without danger of confusion.

The metric $g_{\mu\nu}$ is an invariant tensor, i.e. the components in eq. (1) are the same in every frame. There is a second invariant tensor, the fully antisymmetric Levi-Civita symbol $\epsilon_{\mu\nu\rho\sigma}$. We define

$$\epsilon_{0123} \equiv 1 \, , \tag{2}$$

and all other components follow from antisymmetry.

Exercise 3.1. Show that the contravariant tensor fulfils

$$\epsilon^{0123} = -1 \, . \tag{3}$$

Raising all four indices using the inverse metric induces a minus sign.

For both physics and formal reasons, it is sometimes interesting to work in space-times with $d \neq 4$. In d dimensions, we can define a Levi-Civita symbol with d indices $\epsilon_{\mu_1\mu_2\ldots\mu_d}$. This tensor fulfils the relation

$$\epsilon^{\mu_1\mu_2\ldots\mu_d} \epsilon_{\nu_1\nu_2\ldots\nu_d} = (-1)^{d-1} \begin{vmatrix} \delta^{\mu_1}_{\nu_1} & \ldots & \delta^{\mu_1}_{\nu_d} \\ \vdots & & \vdots \\ \delta^{\mu_d}_{\nu_1} & \ldots & \delta^{\mu_d}_{\nu_d} \end{vmatrix} . \tag{4}$$

The symbol $|\ldots|$ denotes the determinant of the $d \times d$ matrix. Products of ϵ-tensors can thus always be eliminated in favor of metric tensors. A related, useful relation reads

$$\epsilon_{\nu_1\nu_2\ldots\nu_d} A^{\nu_1}{}_{\mu_1} A^{\nu_2}{}_{\mu_2} \ldots A^{\nu_d}{}_{\mu_d} = \det(A) \, \epsilon_{\mu_1\mu_2\ldots\mu_d} \, . \tag{5}$$

By choosing a proper Lorentz transformation Λ for the matrix A one immediately sees that ϵ is indeed an invariant tensor.

Exercise 3.2. Write out and derive eq. (4) in $d = 2$ space-time dimensions.

Exercise 3.3. Derive eq. (4) for d space-time dimensions.

Exercise 3.4. Derive eq. (5).

For $d = 4$, one obtains several useful identities by contracting indices in eq. (4):

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\alpha\beta\gamma} = -\left(\delta^\nu_\alpha\delta^\rho_\beta\delta^\sigma_\gamma + \delta^\nu_\beta\delta^\rho_\gamma\delta^\sigma_\alpha + \delta^\nu_\gamma\delta^\rho_\alpha\delta^\sigma_\beta - \delta^\nu_\beta\delta^\rho_\alpha\delta^\sigma_\gamma - \delta^\nu_\alpha\delta^\rho_\gamma\delta^\sigma_\beta - \delta^\nu_\gamma\delta^\rho_\beta\delta^\sigma_\alpha\right) , \tag{6}$$

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\alpha\beta} = -2\left(\delta^\rho_\alpha\delta^\sigma_\beta - \delta^\rho_\beta\delta^\sigma_\alpha\right) , \tag{7}$$

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\alpha} = -6\,\delta^\sigma_\alpha , \tag{8}$$

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma} = -24 . \tag{9}$$

Exercise 3.5. Derive two of these four relations.

3.2 Dirac Algebra

The Dirac equation is formulated using complex $n \times n$ matrices γ^μ (called the Dirac or simply the γ-matrices) which fulfil

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\,g^{\mu\nu}\,\mathbb{1} , \tag{10}$$

where $\mathbb{1}$ is the $n \times n$ identity matrix. The algebra generated by these matrices (i.e. sums of products of matrices with complex coefficients), defines the Clifford (or Dirac) algebra.

One reason that the Dirac matrices are useful, is that they allow one to easily obtain representations of the Lorentz group for particles with half-integer spin. Indeed, the matrix

$$S^{\mu\nu} \equiv \frac{i}{4}\,[\gamma^\mu, \gamma^\nu] \tag{11}$$

fulfils the commutation relations of the Lorentz algebra, which read

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\right) , \tag{12}$$

where $J^{\mu\nu}$ are the generators of the Lorentz transformations. Due to antisymmetry under $\mu \leftrightarrow \nu$, there are six generators $J^{\mu\nu}$, which contain the boosts J^{0i} and the rotations J^{ij}.

Exercise 3.6. Show that $J^{\mu\nu} = S^{\mu\nu}$ fulfils the commutations relations (12). To see this, verify first that

$$[S^{\mu\nu}, \gamma^\rho] = i\gamma^\mu g^{\nu\rho} - i\gamma^\nu g^{\mu\rho} \tag{13}$$

and then use the Jacobi identity

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0 \tag{14}$$

with $A = S^{\mu\nu}$, $B = \gamma^\rho$, $C = \gamma^\sigma$.

The Dirac algebra can be introduced in different space-time dimensions and both for the Minkowski metric in eq. (1) and for an ordinary Euclidean metric $g^{ij} = \delta^{ij}$. Indeed, for three Euclidean dimensions, the Pauli matrices generate the Clifford algebra,

$$\{\sigma^i, \sigma^j\} = 2\,\delta^{ij}\,\mathbb{1}\,, \tag{15}$$

with $n = 2$. For $d = 4$ space-time dimensions 2×2 matrices are not sufficient. The reason is that an arbitrary 2×2 matrix A can be written as a linear combination $A = c_0 \mathbb{1} + c_i\,\sigma_i$ so that it is not possible to find a fourth anti-commuting matrix. The lowest-dimensional matrices which can realize the $d = 4$ algebra have $n = 4$.

Exercise 3.7. Find an explicit representation of the algebra for $n = 4$. Consider matrices which have blocks of σ_i, $\mathbb{1}$ or 0 as 2×2 submatrices. (Obviously, solutions are in every field-theory book, but it should be fun to find a representation yourself.)

Exercise 3.8. Show that if γ^μ is a representation, then also $\tilde{\gamma}^\mu \equiv S\gamma^\mu S^{-1}$ is one, for an arbitrary invertible $n \times n$ matrix S.

Also the opposite statement is true (but more difficult to show): any representations $\tilde{\gamma}^\mu$ and γ^μ can be related with a suitable matrix S. The representations are thus equivalent up to a change of basis in the vector space on which they act. Many computations, in particular all exercises in the rest of this section, can be performed without relying on an explicit representation of the γ matrices.

Exercise 3.9. Using only the properties in eq. (10), derive the following expressions:

$$\gamma^\mu \gamma_\mu = d\,\mathbb{1}\,, \tag{16}$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = -(d-2)\,\gamma^\nu\,, \tag{17}$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = (d-2)\gamma^\nu\gamma^\rho + 2\gamma^\rho\gamma^\nu\,, \tag{18}$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu + (4-d)\gamma^\nu\gamma^\rho\gamma^\sigma\,. \tag{19}$$

Simplify eq. (18) for $d = 4$ dimensions.

In many QFT computations one needs to compute traces of products of γ-matrices. These can be evaluated using an explicit representation, but it is more elegant to compute them in a representation-independent way, using eq. (10). Let us first compute the trace of a single matrix. Consider

$$(2-d)\mathrm{tr}(\gamma^\nu) = \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma_\mu) = \mathrm{tr}(\gamma^\mu\gamma_\mu\gamma^\nu) = d\,\mathrm{tr}(\gamma^\nu)\,. \tag{20}$$

It follows that the trace must vanish for $d \neq 1$. Similarly, one can show that the trace of any odd number of Dirac matrices vanishes.

Exercise 3.10. Show that $\mathrm{tr}(\gamma^\nu\gamma^\rho\gamma^\sigma) = 0$.

Next, consider the trace of two matrices

$$\mathrm{tr}(\gamma^\mu\gamma^\nu) = \frac{1}{2}\mathrm{tr}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = g^{\mu\nu}\,\mathrm{tr}(\mathbb{1})\,. \tag{21}$$

There is a simple algorithm to compute any trace along these lines. Consider the trace of the product of m matrices and use the cyclicity of the trace to rewrite

$$2\,\mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_m}) = \mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_m}) + \mathrm{tr}(\gamma^{\mu_m}\gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_{m-1}})\;. \tag{22}$$

In a second step one now uses eq. (10) to anti-commute γ^{μ_m} in the second term back to the right-most position. One obtains commutator terms containing fewer γ matrices and finally the term $\mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_m})$, but with a minus sign since one had to anticommute $(m-1)$ times. Because of the sign, this term cancels the first term on the right-hand side of eq. (22). The end result is that the trace of m matrices gets reduced to sum of traces with $(m-2)$ matrices.

Exercise 3.11. Use this algorithm to show that

$$\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = (g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})\,\mathrm{tr}(\mathbb{1})\;, \tag{23}$$

$$\begin{aligned}\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\alpha\gamma^\beta) &= g^{\mu\nu}\mathrm{tr}(\gamma^\rho\gamma^\sigma\gamma^\alpha\gamma^\beta) - g^{\mu\rho}\mathrm{tr}(\gamma^\nu\gamma^\sigma\gamma^\alpha\gamma^\beta) + g^{\mu\sigma}\mathrm{tr}(\gamma^\nu\gamma^\rho\gamma^\alpha\gamma^\beta)\\ &\quad - g^{\mu\alpha}\mathrm{tr}(\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\beta) + g^{\mu\beta}\mathrm{tr}(\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\alpha)\;.\end{aligned} \tag{24}$$

Let us now consider $d = 4$ dimensions. The CPT theorem states that every reasonable QFT must be invariant under a combination of charge conjugation (C), parity (P) and time reversal (T), but not necessarily under each of these transformations separately. One can show that in the Weyl basis (solution of Exercise 3.7 where γ^0 is not diagonal), these transformations are represented by $\Gamma_P = \gamma^0$, $\Gamma_T = \gamma^1\gamma^3$ and $\Gamma_C = i\gamma^2$. The product of these matrices is thus $\Gamma_{CPT} = \Gamma_T\Gamma_P\Gamma_C = \mathrm{i}\gamma^1\gamma^3\gamma^0\gamma^2 = -i\gamma^0\gamma^1\gamma^2\gamma^3$. It is therefore useful to identify this particular combination with the additional matrix γ_5,

$$\gamma_5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3 = i\gamma_0\gamma_1\gamma_2\gamma_3\;. \tag{25}$$

Exercise 3.12. Show that this matrix fulfils

$$\{\gamma_5, \gamma^\mu\} = 0\;, \qquad (\gamma_5)^2 = \mathbb{1}\;, \qquad \gamma_5 = -\frac{i}{24}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\,. \tag{26}$$

Exercise 3.13. Verify the following trace relations:

$$\mathrm{tr}(\gamma_5) = 0\,, \tag{27}$$

$$\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma_5) = 0\,, \tag{28}$$

$$\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5) = -4i\epsilon^{\mu\nu\rho\sigma}\,. \tag{29}$$

Exercise 3.14. Because traces of gamma matrices are omnipresent in High Energy Physics, it is interesting to investigate how they can be evaluated most efficiently.
Consider the following quantity:

$$T = p_1^{\nu_1} g_{\nu_1,\mu_1} \cdots p_n^{\nu_n} g_{\nu_n,\mu_n} \mathrm{tr}\left(\gamma^{\mu_1} \cdots \gamma^{\mu_n}\right) \tag{30}$$

involving n arbitrary four-momenta $p_i^{mu_i}$. Compare the complexity in n of its evaluation in terms of the number of multiplications for the following two strategies:

- a) By first using the result of Exercise 3.10 in order to evaluate the trace in terms of only scalar products of the momenta.
- b) By direct numerical evaluation of the trace using explicit matrix multiplication with the sparse matrix representation found at Exercise 3.7, that is:

$$T = \sum_{i_0=0,3} \left[\left(\sum_{m_1=0,3} (-1+2\delta_{0,m_1}) p_1^{m_1} \gamma^{m_1} \right)_{i_0 i_1} \cdots \left(\sum_{m_n=0,3} (-1+2\delta_{0,m_n}) p_n^{m_n} \gamma^{m_n} \right)_{i_{2(n-1)} i_0} \right] \tag{31}$$

OPTIONAL: For those of you most intrigued by the above, you implement the two strategies in an optimized computer code and compare the time needed to evaluate the trace for a given number of momenta, so as to verify the theoretical complexity analysis and identify the approximate value n for which one strategy overtakes the other.