4 Force Carriers

The spin-1 fields (vector bosons) are ubiquitous in gauge theories such as the Standard Model. They are the force carriers, eg. photon in QED, gluons in QCD and $W^\pm$ and Z bosons in the nuclear weak force. The vector fields can be either massive ($W^\pm$ and Z) or massless (gluons, photon). The mass origin of a massive vector field is a fundamental question in particle physics.

4.1 Maxwell equations

The Maxwell equations (in Lorentz-Heaviside units) are

$$\nabla \cdot \vec{E} = \rho\,, \quad \nabla \times \vec{B} - \partial_t \vec{E} = \vec{j}\ , \tag{1}$$

$$\nabla \cdot \vec{B} = 0\,, \quad \nabla \times \vec{E} + \partial_t \vec{B} = 0\ . \tag{2}$$

1. Show that the homogeneous equations (2) are automatically satisfied if the fields are written in terms of the real 4-vector potential $A^\mu = (\phi, A^i)$, with

$$\vec{A} = \sum_i A^i \vec{e}_i, \quad \vec{B} = \nabla \times \vec{A}, \quad \vec{E} = -\partial_t \vec{A} - \nabla\phi\ . \tag{3}$$

2. Defining the antisymmetric field strength tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu\ , \tag{4}$$

and the real current vector $j^\mu = (\rho, \vec{j})$, show that

$$E^i = -F^{0i}\,, \quad B^i = -\frac{1}{2}\epsilon^{ijk} F_{jk}\ , \tag{5}$$

and that the inhomogeneous equations (1) can be compactly written as

$$\partial_\mu F^{\mu\nu} = j^\nu\ . \tag{6}$$

3. Using the principle of least action, derive the classical equation of motion (6) for the field A^μ from the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu\ . \tag{7}$$

4.2 Massive vector field

Let us add the mass term in the above Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu + \frac{m^2}{2} A_\mu A^\mu\ , \tag{8}$$

to describe the propagation of a massive spin-1 field.

1. Derive the classical equation of motion for the massive spin-1 field, called the Proca equation.

2. Show that the Proca equation implies the consistency relation

$$m^2 \partial_\mu A^\mu = \partial_\mu j^\mu \ . \tag{9}$$

For a conserved (Noether) current $\partial_\mu j^\mu = 0$ and non-zero mass m, this imposes the condition $\partial_\mu A^\mu = 0$. This condition implies that not all four components of the field A_μ are independent. (The real spin-1 field has three degrees of freedom, two transverse and one longitudinal polarisation, as we prove later.)

3. Using the above condition, show that the Proca equation simplifies to

$$(\Box + m^2) A_\mu(x) = j_\mu(x) \ , \tag{10}$$

where $\Box = \partial_\mu \partial^\mu$.

For a free-field theory ($j_\mu = 0$) every component of the field satisfies the Klein-Gordon equation. We can write the solutions as a superposition of plane waves

$$A_\mu(x) = \sum_{\lambda=0}^{3} \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3 2\omega_k} \left[\epsilon_\mu(\vec{k},\lambda) a(\vec{k},\lambda) e^{-ik\cdot x} + \epsilon^*_\mu(\vec{k},\lambda) a^*(\vec{k},\lambda) e^{ik\cdot x} \right] \ , \tag{11}$$

where $k^0 \equiv \omega_k = \sqrt{m^2 + |\vec{k}|^2}$.

The four auxiliary polarization vectors $\epsilon_\mu(\vec{k},\lambda)$ can be chosen such that they form an orthonormal basis of Minkowski space-time:

$$\epsilon^*_\mu(\vec{k},\lambda) \epsilon^\mu(\vec{k},\lambda') = g_{\lambda\lambda'} \ . \tag{12}$$

In principle, these vectors can be chosen real, but sometimes it is convenient to work with complex ones (circular polarizations). The first two of the vectors (the transverse polarizations) are usually chosen to have the form

$$\epsilon^\mu(\vec{k},1) = (0, \vec{\epsilon}(\vec{k},1)) \ , \quad \epsilon^\mu(\vec{k},2) = (0, \vec{\epsilon}(\vec{k},2)) \ , \tag{13}$$

with

$$k \cdot \epsilon(\vec{k},1) = k \cdot \epsilon(\vec{k},2) = 0 \ , \tag{14}$$

and

$$\vec{\epsilon}^{\,*}(\vec{k},i) \cdot \vec{\epsilon}(\vec{k},j) = \delta_{ij} \ . \tag{15}$$

1. The third space-like polarization vector is chosen to have its three-vector parallel to $\vec{k}$. This longitudinal polarization vector has the form

$$\epsilon^\mu(\vec{k},3) = (A, B\vec{k}) \ , \tag{16}$$

where the parameters A and B are chosen such that it is orthogonal to k,

$$k \cdot \epsilon(\vec{k},3) = 0 \ , \tag{17}$$

and normalised according to equation (12). Determine the coefficients A and B as a function of k.

2. The final polarization vector points along the direction of k_μ by equation (12), that is $\epsilon_\mu(\vec{k}, 0) = C k_\mu$. Determine the coefficient C.

3. Show that the condition $\partial_\mu A^\mu = 0$ implies vanishing annihilation operator $a(\vec{k}, 0) = 0$ in equation (11). That is, the massive field thus contains three independent solutions ("polarizations") for a given momentum.

4. Show that the four polarization vectors fulfil the completeness relation

$$\sum_{\lambda,\lambda'=0}^{3} g_{\lambda\lambda'} \epsilon^*_\mu(\vec{k}, \lambda) \epsilon_\nu(\vec{k}, \lambda') = g_{\mu\nu} \; . \tag{18}$$

5. Derive from this that the three physical polarization vectors fulfil the completeness relation

$$\sum_{\lambda=1}^{3} \epsilon^*_\mu(\vec{k}, \lambda) \epsilon_\nu(\vec{k}, \lambda) = -g_{\mu\nu} + \frac{k_\mu k_\nu}{m^2} \; . \tag{19}$$

The sum of the physical polarisation can be related to the vector boson Feynman propagator in the *unitary gauge*,

$$D_F^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i\left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}\right)}{k^2 - m^2 + i\epsilon} e^{-ik(x-y)} \; . \tag{20}$$

Please note the factor of m^{-2}, which makes the limit $m \to 0$ nontrivial.

4.3 $U(1)$ gauge invariance

For $m = 0$ and assuming current conservation ($\partial_\mu j^\mu = 0$) the Lagrangian in equation (8) is invariant under gauge transformations

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x) \; , \tag{21}$$

where $\alpha(x)$ an arbitrary differentiable scalar field.

1. Show that these transformations leave $F_{\mu\nu}$ invariant and therefore also the electric and magnetic fields unchanged.

2. Use integration by parts to show the invariance of the action for a conserved current $\partial_\mu j^\mu = 0$.

3. Show that for any given field A_μ one can always make a gauge transformation such that the condition $\partial_\mu A^\mu = 0$ is fulfilled. This is called Lorenz gauge (since the imposed condition is Lorentz invariant).

4. The condition $\partial_\mu A^\mu = 0$ eliminates one unphysical degree of freedom. However, even imposing $\partial_\mu A^\mu = 0$ does not fix the gauge freedom completely, show that we can still perform gauge transformations which fulfil the scalar wave equation

$$\Box \alpha(x) = 0 \ . \tag{22}$$

Such transformations correspond to another unphysical degree of freedom so we conclude that the photon field (massless vector boson) only contains two physical degrees of freedom.

5. The Proca action for a massive spin-1 field is not gauge invariant, but the Stueckelberg action

$$\mathcal{L}(A, \phi) = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{m^2}{2} \left(\partial_\mu \phi + A_\mu\right) \left(\partial^\mu \phi + A^\mu\right) - j^\mu A_\mu \ , \tag{23}$$

which involves an additional real scalar field ϕ can be invariant under the simultaneous gauge transformations of $A_\mu(x)$ and $\phi(x)$. Find the appropriate transformation of $\phi(x)$ given equation (21).

6. Choose the gauge condition for $\phi(x)$, such that the Stueckelberg action reduces to the Proca action.

One can thus view a massive spin-1 field as a combination of a gauge invariant field A_μ, with two degrees of freedom, and an additional scalar field ϕ, which provides the third degree of freedom (the longitudinal polarization). An explicit realization of this is the Higgs mechanism, where additional scalar fields (the Goldstone bosons) provide a mass term for the gauge fields of the weak interactions.