6 Dirac Lagrangian and the Dirac equation

The Dirac equation, first written by British physicist Paul Dirac in 1928, offers a unified quantum and relativistic description of spin-$\frac{1}{2}$ fermions (quarks and leptons). You will solve this equation in this exercise class and, in doing so, also stumble upon puzzling negative energy solutions that Dirac himself did not know how to interpret. It was only four years later, in 1932, that Carl Anderson discovered the positron which naturally explained the negative energy solutions as anti-particles!

Starting from the Lorentz-invariant Dirac Lagrangian constructed last week,

$$\mathcal{L}(x) = \bar{\psi}(x) i\gamma^\mu \partial_\mu \psi(x) - m\bar{\psi}(x)\psi(x) \ , \tag{1}$$

show that:

1. The action $S = \int d^4x\, \mathcal{L}(x)$ is real provided that the fields vanish fast enough at infinity. The mass m is a real parameter.

2. Derive the Dirac equation from the principle of least action. (To get the equations of motion one can consider independent variations of the real and imaginary parts of the field $\psi(x)$. A shortcut leading to the same result is to consider independent variations of ψ and $\bar{\psi}$.)

3. Multiply the Dirac equation by $(i\gamma^\mu\partial_\mu + m)$ from the left and show that each component of the spinor field fulfils the Klein-Gordon equation

$$(\Box + m^2)\psi(x) = 0 \ . \tag{2}$$

6.1 Plane-wave solutions of the Dirac equation

Similar to the Klein-Gordon equation, we will find plane-wave solutions to the Dirac equation. Let us first consider a plane wave of the form

$$\psi(x) = u(\vec{k})e^{-ikx} \ , \tag{3}$$

where $kx \equiv k_\mu x^\mu$ with $k^0 = \omega_k$ so that it fulfils the Klein-Gordon equation (on-shell dispersion relation). In the following, we will use the chiral representation of the γ^μ matrices:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \ , \tag{4}$$

where

$$\sigma^\mu = (\mathbb{1}, \sigma^i) \ , \quad \bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i) \ . \tag{5}$$

1. Show that the plane wave is a solution of the Dirac equation if the spinor $u(\vec{k})$ fulfils the matrix equation

$$(\not{k} - m)u(\vec{k}) = 0 \ , \tag{6}$$

where $\not{k} = \gamma^\mu k_\mu$.

2. As for the case of the Klein-Gordon equation, we also have "negative-frequency" solutions

$$\psi(x) = v(\vec{k})e^{+ikx} \ . \tag{7}$$

Show that for these (which describe anti-particles) the spinor $v(\vec{k})$ fulfils the equation

$$(\not{k} + m)v(\vec{k}) = 0 \ . \tag{8}$$

3. Consider the positive-frequency solutions with vanishing three-momentum $\vec{k} = \vec{0}$ and show that the spinor

$$u(0) = \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix} \ , \tag{9}$$

fulfils the equation for an arbitrary two-component spinor ξ.

4. According to Lorentz transformation for the spinor fields studied last week, the two-component spinor transforms under rotations as

$$\xi \to e^{i\theta_i \frac{\sigma^i}{2}} \xi \ . \tag{10}$$

This is the familiar transformation law of quantum mechanical two-component spinors who describe two states of a spin-1/2 particle. The spinor $\xi = (1,0)^T$ fulfils

$$\frac{\sigma_3}{2}\xi = +\frac{1}{2}\xi \ , \tag{11}$$

and describes a particle with spin along the positive z direction, whereas $\xi = (0,1)^T$ has spin in the negative z direction.

Our work on the Lorentz transformations of the spinors now pays off. The solutions for particles with momentum $\vec{k}$ can be obtained by performing a boost of the solution for the particle at rest. We have already derived the explicit form of the boost last time,

$$D(\Lambda) = \begin{pmatrix} \exp\left(-\beta_i\sigma^i/2\right) & 0 \\ 0 & \exp\left(\beta_i\sigma^i/2\right) \end{pmatrix} \ , \tag{12}$$

and will now apply it to the spinor in eq. (9).

   (a) Consider a boost along the third direction $\omega_{03} = -\omega_{30} = \beta$. Determine the value of β needed to arrive at a boost of the form

$$k^\mu = (m,0,0,0) \to k'^\mu = \Lambda^\mu_{\ \nu}k^\nu = (E,0,0,k^3) \qquad E = \omega_{k^3} = \sqrt{(k^3)^2+m^2} \ . \tag{13}$$

   [Hint: Use the Lorentz transformations for the vector representation $\Lambda = \exp\left(-\frac{i}{2}\omega_{\alpha\beta}J^{\alpha\beta}\right)$ where $\left(J^{\alpha\beta}\right)^\mu_{\ \nu} = i\left(g^{\mu\alpha}\delta^\beta_\nu - g^{\mu\beta}\delta^\alpha_\nu\right)$. Since the other components do not play a role, it is convenient to write the vectors in the form $k^\mu = (\omega_{k^3}, k^3)$ to keep the notation compact.]

(b) Use the obtained value of the boost parameter to boost the spinor $u(0)$ via eq. (12) for the case $\xi = (1,0)^T$ so that it has three momentum $\vec{k} = (0,0,k^3)$.

(c) The same exercise for the case $\xi = (0,1)^T$.

(d) It turns out that the general boosted solution can be written in the elegant form

$$u(\vec{k}) = \begin{pmatrix} \sqrt{k_\mu \sigma^\mu}\,\xi \\ \sqrt{k_\mu \bar{\sigma}^\mu}\,\xi \end{pmatrix} , \tag{14}$$

where the square root of the matrix is obtained by first transforming to the diagonal basis, taking the square root of the eigenvalues and transforming back.
Derive the matrix relation

$$k_\mu \sigma^\mu \, k_\nu \bar{\sigma}^\nu = k^2 = m^2 , \tag{15}$$

and use it to verify that eq. (14) satisfies the Dirac equation.
Analogously, the general boosted solution for the anti-particle spinors can be written as

$$v(\vec{k}) = \begin{pmatrix} \sqrt{k_\mu \sigma^\mu}\,\eta \\ -\sqrt{k_\mu \bar{\sigma}^\mu}\,\eta \end{pmatrix} . \tag{16}$$

6.2 Completeness relation

It is convenient to introduce orthonormal basis spinors ξ_s (with $s = 1,2$) which fulfil

$$\xi_r^\dagger \xi_s = \delta_{rs} , \tag{17}$$

and then to denote the solution for the corresponding spinor by $u_s(\vec{k})$. The two-component spinors then fulfil the completeness relation

$$\sum_{s=1,2} \xi_s \xi_s^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{18}$$

Similar for η.

1. Using the explicit result in eq. (14), show that the spinors are normalized as

$$u_s^\dagger(\vec{k}) u_s(\vec{k}) = 2\omega_k , \tag{19}$$

$$\bar{u}_s(\vec{k}) u_s(\vec{k}) = 2m . \tag{20}$$

2. Finally prove that

$$\sum_{s=1,2} u_s(\vec{k}) \bar{u}_s(\vec{k}) = \not{k} + m , \tag{21}$$

$$\sum_{s=1,2} v_s(\vec{k}) \bar{v}_s(\vec{k}) = \not{k} - m . \tag{22}$$