7 Free particle propagator

In the first exercise of this class, we had studied Gaussian intergrals and derived the structure of Wick's theorem for N-point correlators and introduced the notion of a generating functional. The generalisation to a quantum scalar field theory yields:

$$\langle \boldsymbol{T}\left\{\phi(x_1)\phi(x_2)...\phi(x_N)\right\}\rangle \equiv \frac{\int \mathcal{D}\phi\ e^{iS[\phi]}\boldsymbol{T}\left\{\phi(x_1)\phi(x_2)...\phi(x_N)\right\}}{\int \mathcal{D}\phi\ e^{iS[\phi]}}\ , \tag{1}$$

where $S[\phi] = \int d^4x\ \mathcal{L}(x)$ and the time-ordering operator $\boldsymbol{T}$ orders the fields from left to right in descending order of the time component of the position at which they are evaluated. The generating functional of the correlation functions is given by

$$Z[J] = \int \mathcal{D}\phi\ e^{iS[\phi]+i\int d^4x\, J(x)\phi(x)}\ , \tag{2}$$

where $J(x)$ is called an external source.

The traditional derivatives $\partial_{b_{i_j}}$ we had used in the first exercise are now replaced by functional derivatives, defined as $\frac{\delta}{\delta J(x)}J(y) = \delta^4(x-y)$, resulting in an analoguous expression for the correlator of N scalar quantum fields:

$$\langle\phi(x_1)\phi(x_2)...\phi(x_N)\rangle = (-i)^N\frac{1}{Z[0]}\ \frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}...\frac{\delta}{\delta J(x_N)}Z[J]\bigg|_{J=0}\ . \tag{3}$$

We will now consider the free theory of a real scalar field $\phi(x)$ with the Lagrangian:

$$\mathcal{L}(x) = -\frac{1}{2}\phi(x)(\Box + m^2)\phi(x)\ , \tag{4}$$

where m is a real parameter and $\Box = \partial_\mu\partial^\mu$. Before computing the free scalar propagator given by $\langle\phi(x_1)\phi(x_2)\rangle$, let us study the generating functional $Z[J]$ in more details.

1. Show that the path integral in eq. (2) converges when considering the small deformation to the mass $m^2 \to m^2 - i\epsilon$. Note the importance of the sign of this complex deformation.

2. Let $iD_F(x-y)$ be the inverse of the operator $(\Box + m^2 - i\epsilon)$, that is

$$(\Box + m^2 - i\epsilon)\ iD_F(x-y) = \delta^4(x-y)\ . \tag{5}$$

In the first class, we had established the following result for Gaussian integrals:

$$\mathcal{N}\int \mathrm{d}^n x\ \exp\left(-\frac{1}{2}x^TMx + a^Tx\right) = \sqrt{\frac{(2\pi)^n}{\det M}}\exp\left(\frac{1}{2}a^TM^{-1}a\right)\ . \tag{6}$$

In complete analogy with the derivation of the above result, show that

$$Z[J] = Z[0]e^{-\frac{1}{2}\int d^4x\, d^4y\ \ J(x)D_F(x-y)J(y)}\ . \tag{7}$$

Hint: Consider "completing the square" using the following field shift by a constant factor:

$$\phi(x) \to \psi(x) = \phi(x) - \int \mathrm{d}y\ iD_F(x-y)J(y) \tag{8}$$

3. We are finally ready to use eq. (3) to compute the Feynman propagator of this real scalar field. Show that it is given by:

$$\langle \boldsymbol{T}\left\{\phi(x_1)\phi(x_2)\right\}\rangle = D_F(x_1 - x_2)\ . \tag{9}$$

4. The propagator is rather straight-forward to obtain in momentum space, where the derivative operators in D_F become momenta, and thus $D_F = k^2 - m^2 + i\varepsilon$. It is however possible to obtain an expression for $D_F(x_1 - x_2)$ by applying the inverse Fourrier transform to the propagator expression in momentum space:

$$D_F(x-y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} e^{-ik\cdot(x-y)}\ . \tag{10}$$

By explicitly integrating the energy component of the momentum integral using Cauchy's theorem, show that

$$D_F(x-y) = \Theta(x^0 - y^0)\Delta(x,y) + \Theta(y^0 - x^0)\Delta(y,x)\ , \tag{11}$$

with $\Delta(x,y)$ defined as

$$\Delta(x,y) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} e^{+i\omega_{\vec{k}}(x^0-y^0) - i\vec{k}\cdot(\vec{x}-\vec{y})}\ , \tag{12}$$

with $\omega_{\vec{k}}$ the dispersion relation $\omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$. Notice the crucial role that the sign of the $+i\epsilon$ prescription plays in this construction and therefore its importance for the causal structure of the theory.

8 Feynman rules

Information on a quantum field theory can be extracted from the correlation functions of the fields. As shown by time-dependent perturbation theory in Quantum Mechanics (cf. discussion of the interaction / Dirac picture), of particular interest are time-ordered correlation functions

$$\langle \Omega \left| \boldsymbol{T}\left\{\phi\left(x_1\right)\ldots\phi\left(x_n\right)\right\}\right| \Omega\rangle\ . \tag{13}$$

From the two-point function, one can obtain information about the spectrum of the theory, whereas the higher-point functions yield the scattering amplitudes. Interacting theories are in general too complicated to allow for an exact computation of these correlation functions, but we can simplify the problem by treating the interaction as a perturbation. For ϕ^4-theory, for example, we write the Lagrangian as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\ , \tag{14}$$

and expand all observables as a power series in the coupling constant λ.

The prescription to compute correlation functions in perturbation theory is quite simple. One computes

$$\langle \Omega \left| \boldsymbol{T} \left\{ \phi\left(x_1\right) \ldots \phi\left(x_n\right) \right\} \right| \Omega \rangle = \frac{1}{Z} \left\langle 0 \left| T \left\{ \phi\left(x_1\right) \ldots \phi\left(x_n\right) \exp\left[i \int \mathrm{d}^4 z \mathcal{L}_{\text{int}}\left(z\right) \right] \right\} \right| 0 \right\rangle \ , \tag{15}$$

where the correlation function on the right-hand side is computed in the free theory. The normalization factor

$$Z = \left\langle 0 \left| \boldsymbol{T} \left\{ \exp\left[i \int \mathrm{d}^4 z \mathcal{L}_{\text{int}}\left(z\right) \right] \right\} \right| 0 \right\rangle \ , \tag{16}$$

arises because the vacuum of the free theory is not the same as the vacuum of the interacting theory. For perturbation theory, the factor

$$\exp\left[i \int \mathrm{d}^4 z \mathcal{L}_{\text{int}}\left(z\right) \right] = 1 + i \int \mathrm{d}^4 z \mathcal{L}_{\text{int}}\left(z\right) - \frac{1}{2} \int \mathrm{d}^4 z' \int \mathrm{d}^4 z \mathcal{L}_{\text{int}}\left(z\right) \mathcal{L}_{\text{int}}\left(z'\right) + \ldots \ , \tag{17}$$

is expanded and the higher-order terms are suppressed by higher powers of the coupling constant. The interaction terms in the theory are polynomials in the fields and so the entire computation reduces to the evaluation of correlation functions in the free theory,

$$\langle 0 \left| \boldsymbol{T} \left\{ \phi\left(x_1\right) \ldots \phi\left(x_n\right) \right\} \right| 0 \rangle \ , \tag{18}$$

where some of the fields arise from the above expansion. Note that this is an equivalent formulation of the path integral formalism used in the previous section.

8.1 Feynman rules for ϕ^4

An important result for the evaluation of correlation functions in the free theory is Wick's theorem. It states that

$$\langle 0 \left| \boldsymbol{T} \left\{ \phi\left(x_1\right) \ldots \phi\left(x_n\right) \right\} \right| 0 \rangle = \sum_{\text{pairings}} G_F\left(x_{i_1} - x_{i_2}\right) G_F\left(x_{i_2} - x_{i_3}\right) \ldots G_F\left(x_{i_{n-1}} - x_{i_n}\right) \ , \tag{19}$$

where the Feynman propagator G_F, is the two-point Green's function that we have shown to be equal to the propagator D_F we obtained in eq. (10).

1. Compute the four-point correlation function

$$\langle \Omega \left| \boldsymbol{T} \left\{ \phi\left(x_1\right) \phi\left(x_2\right) \phi\left(x_3\right) \phi\left(x_4\right) \right\} \right| \Omega \rangle \ , \tag{20}$$

   at first order in λ in ϕ^4-theory. The number of contractions is quite large, but many of them are equivalent because the fields in the interaction Lagrangian live at the same point. It is sufficient to write down the expression, you do not need to perform the integration over the position of the interaction vertex. Represent the contractions graphically by drawing the propagator as a line from x to y.

2. Draw the diagrams for both the numerator in eq. (15) and also the normalization factor Z in eq. (16). Which contributions cancel against the normalisation factor?

3. Compute the Fourier transform of the fully-connected part of the correlator (20). The fully-connected piece is the most relevant, since it contains the scattering amplitude. (Fully-connected: All external lines are connected to each other and there are no vacuum bubbles.)

4. As a shortcut to Wick's theorem, the results for the correlation functions can be obtained using the so-called Feynman rules. For the case of ϕ^4-theory, the momentum-space rules for the connected part of the n-point correlation function at mth order in λ are derived in e.g. *Peskin - Chapter 4*. One first draws all possible fully-connected diagrams with n external legs and m interaction vertices. Then convert each diagram into the mathematical expression using the Feynman rules.

   Identify what are the Feynman rules for ϕ^4 theory in the previous computation of (20). More specifically, explains what mathematical expression needs to be inserted for each vertex and propagator, both in position space and momentum space.

8.2 Feynman rules for QED

The QED Lagrangian can be written as

$$\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i \not{D} - m)\psi \,, \tag{21}$$

with the covariant derivative defined as $D_\mu = \partial_\mu + ieA_\mu$.

The Feynman rules for QED are similar to the scalar case. Wick's theorem applies also for vector and fermion fields (except that one has to be careful about signs with anti-commuting fermion fields), but we now need expressions for the fermion and photon propagators. In the so-called Feynman gauge the photon propagator reads

$$G_F^{\mu\nu}(x-y) = \langle 0 \left| \boldsymbol{T} \left[A^\mu(x) A^\nu(y) \right] \right| 0 \rangle = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 + i\epsilon} \left(-g_{\mu\nu} \right) e^{-ik\cdot(x-y)} \,. \tag{22}$$

The numerator $-g_{\mu\nu}$ arises from the sum over polarizations and includes unphysical degrees of freedom, but one can show that the unphysical polarizations do not contribute to physical quantities. The photon propagator is not unique but depends on the gauge choice; furthermore, so-called Faddeev-Popov ghosts need to be introduced in general.

The fermion propagator reads

$$S_{F\alpha\beta}(x-y) = \left\langle 0 \left| \boldsymbol{T} \left[\psi_\alpha(x) \bar{\psi}_\beta(y) \right] \right| 0 \right\rangle = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i \left(k_\mu \gamma^\mu_{\alpha\beta} + m \delta_{\alpha\beta} \right)}{k^2 - m^2 + i\epsilon} e^{-ik\cdot(x-y)} \,. \tag{23}$$

Typically, the abbreviation $\not{k} \equiv k_\mu \gamma^\mu$ is used for the numerator and often one does not explicitly write out the Dirac indices α and β. Armed with these expressions and Wick's theorem, we can now evaluate correlation functions in QED.

1. Compute the three-point function

$$\left\langle \Omega \left| T \left\{ \psi_\alpha \left(x_1 \right) A_\mu \left(x_2 \right) \bar{\psi}_\beta \left(x_3 \right) \right\} \right| \Omega \right\rangle \,. \tag{24}$$

Represent the result graphically, using a line with an arrow to represent the fermion propagator and a wiggly line for the photon. Read off the Feynman rule for the QED vertex.

2. The QED Feynman rules in momentum space have the same structure as the ones in the scalar case, but the propagators and vertices now carry Dirac indices α, β and Lorentz indices μ, ν. Think of suitable correlation functions which can allow you to derive QED Feynman rules (in momentum space) using Wick's theorem:

   (a) What is the propagator Feynman rule for a photon and a fermion in the momentum space?

   (b) What is the QED vertex Feynman rule in the momentum space?

   (c) Show that at each vertex one imposes momentum conservation.

   (d) What is the Feynman rule for the closed loop? What one does with the undetermined loop momenta?

   (e) Show that each closed fermion loop includes a factor (-1).