9 Reduction formula and S-matrix

Scattering amplitudes are needed to compute cross sections measured in particle physics experiments. We have already discussed time-ordered vacuum correlation functions in perturbation theory. The scattering amplitudes can be extracted from these using the Lehmann-Symanzik-Zimmermann (LSZ) reduction, but a thorough derivation of this relation is beyond the scope of these exercises and also difficult to find in the literature general.

Let us briefly sketch how the reduction formula is obtained by considering the four-point correlation function. To extract the contribution of an outgoing particle generated with field $\phi(x_1)$ one considers the Fourier transform with respect to x_1,

$$F(p_1, x_2, x_3, x_4) = \int d^4x_1\, e^{ip_1x_1} \langle \Omega | \boldsymbol{T} \{\phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\} | \Omega \rangle \ . \tag{1}$$

One can show (and this is the subtle part) that if there is a single particle state with mass m, then for $p_1^0 > 0$ the function F develops a pole at $p_1^2 = m^2$, which describes the propagation of an outgoing particle at very large times,

$$F(p_1, x_2, x_3, x_4) = \frac{i}{p_1^2 - m^2} \langle \Omega | \phi(0) | \vec{p}_1; \text{out} \rangle \langle \vec{p}_1; \text{out} | T\{\phi(x_2)\,\phi(x_3)\,\phi(x_4)\} | \Omega \rangle + \dots \ . \tag{2}$$

The dots represent contributions from other states, which do not give rise to a pole. Now, the Feynman diagrams in momentum space do involve poles coming from the external propagators. The expectation value

$$\langle \Omega | \phi(0) | \vec{p}_1 \rangle \equiv \sqrt{Z} = 1 + \mathcal{O}(\lambda) \ , \tag{3}$$

is called the on-shell wave-function renormalization constant. It can be computed from the two-point function (see KL spectral representation).

We can repeat the procedure for the other three fields to get

$$\int d^4x_1\, e^{ip_1x_1} \int d^4x_2\, e^{ip_2x_2} \int d^4x_3\, e^{-iq_Ax_3} \int d^4x_4\, e^{-iq_Bx_4} \langle \Omega | \boldsymbol{T} \{\phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\} | \Omega \rangle = \tag{4}$$

$$= \langle \vec{p}_1, \vec{p}_2; \text{out} | \vec{q}_A, \vec{q}_B; \text{in} \rangle \frac{i\sqrt{Z}}{p_1^2 - m^2} \frac{i\sqrt{Z}}{p_2^2 - m^2} \frac{i\sqrt{Z}}{q_A^2 - m^2} \frac{i\sqrt{Z}}{q_B^2 - m^2} \ . \tag{5}$$

To get the $2 \to 2$ scattering amplitude one thus computes the Fourier transform of the four- point function and divides by the external propagators. A Green's function from which the external propagators have been removed is called an amputated Green's function. The quantity

$$S_{fi} = \langle \vec{p}_1, \vec{p}_2; \text{out} | \vec{q}_A, \vec{q}_B; \text{in} \rangle \tag{6}$$

is called the S-matrix. The non-trivial part of the S-matrix, in which the incoming particles interact, arises from the fully-connected diagrams (all external lines are connected to each other) and is called the scattering matrix M. It is defined as

$$(2\pi)^4 \delta^{(4)}(q_A + q_B - p_1 - p_2) iM(q_A, q_B \to p_1, p_2) \equiv \langle \vec{p}_1, \vec{p}_2; \text{out} | \vec{q}_A, \vec{q}_B; \text{in} \rangle |_{\text{connected}} \ . \tag{7}$$

In practical terms, the computation of M is quite simple. One uses the momentum-space Feynman rules for the connected Green's functions, removes the external propagators and multiplies by the appropriate Z-factor (at lowest order $Z = 1$), and by $(-i)$. For ϕ^4-theory one obtains for the $2 \to 2$ amplitude at the lowest order

$$M = -\lambda \; . \tag{8}$$

For external fermions and photons, the reduction formula has the same structure, but one needs the matrix elements

$$\begin{aligned} \left\langle \Omega \left| \psi_\alpha(0) \right| e^-(\vec{p}, s) \right\rangle &= Z_\psi^{\frac{1}{2}} u_\alpha(\vec{p}, s) \; , \\ \left\langle \Omega \left| \bar{\psi}_\alpha(0) \right| e^+(\vec{p}, s) \right\rangle &= Z_\psi^{\frac{1}{2}} \bar{v}_\alpha(\vec{p}, s) \; , \\ \left\langle \Omega \left| A_\mu(0) \right| \gamma(\vec{p}, \lambda) \right\rangle &= Z_A^{\frac{1}{2}} \epsilon_\mu(\vec{p}, \lambda) \; , \end{aligned} \tag{9}$$

and their complex conjugates. Here u, v are Dirac spinors, and ϵ_μ is a polarization vector.

You can use the plane wave solutions for both fermion and vector fields from previous exercises

$$\psi_\alpha(x) = \sum_{s=\pm} \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[b_s(\vec{k}) u_\alpha(\vec{k}, s) e^{-ikx} + d_s^\dagger(\vec{k}) v_\alpha(\vec{k}, s) e^{ikx} \right] \; , \tag{10}$$

and

$$A_\mu(x) = \sum_{\lambda=0}^{3} \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[a(\vec{k}, \lambda) \varepsilon_\mu(\vec{k}, \lambda) e^{-ikx} + a^*(\vec{k}, \lambda) \varepsilon_\mu^*(\vec{k}, \lambda) e^{ikx} \right] \; . \tag{11}$$

The canonical quantization (anti-)commutation relations for the fermionic and vector creation and annihilation operators (with the above convention for the fields) are

$$\begin{aligned} \left\{ b_s(\vec{k}), b_{s'}^\dagger(\vec{p}) \right\} &= (2\pi)^3 2\omega_k \delta^{(3)}(\vec{k} - \vec{p}) \delta_{ss'} \; , \\ \left[a(\vec{k}, \lambda), a^\dagger(\vec{p}, \lambda') \right] &= (2\pi)^3 2\omega_k \delta^{(3)}(\vec{k} - \vec{p}) \delta_{\lambda\lambda'} \; , \end{aligned} \tag{12}$$

and all the other (anti-)commutation relations are equal to zero.

1. Verify the matrix elements in eq. (9) by using the expressions for the free field operators for spinor and vector fields. They are obtained by replacing the Fourier expansion coefficients by creation and annihilation operators (e.g. $b_s^\dagger(\vec{k})|\Omega\rangle = |e^-(\vec{k}, s)\rangle$). Subsequently one uses canonical commutation relations. The Z-factors for free fields are equal to 1.

One can formulate the above prescription as Feynman rules directly for M. Suppressing the Z-factors, which for our leading-order computations are equal to 1, one first computes the amputated amplitude using the Feynman rules given last time. Then, for the external lines one multiplies with $u(\vec{p}, s)$, $\bar{v}(\vec{p}, s)$ and $\epsilon(\vec{p}, \lambda)$ for incoming and $\bar{u}(\vec{p}, s)$, $v(\vec{p}, s)$ and $\epsilon^*(\vec{p}, \lambda)$ outgoing particles.

10 $e^+e^- \to \mu^+\mu^-$ in QED

Let us compute our first real-life scattering.

1. Show that the scattering amplitude M for the process $e^-(q_A, s_A)e^+(q_B, s_B) \to \mu^-(p_1, r_1)\mu^+(p_2, r_2)$ reads
$$\mathcal{M}^{(e^+e^-\to\mu^+\mu^-)} := M = \frac{ie^2 \bar{v}\left(\vec{q}_B, s_B\right)\gamma^\mu u\left(\vec{q}_A, s_A\right)\bar{u}\left(\vec{p}_1, r_1\right)\gamma_\mu v\left(\vec{p}_2, r_2\right)}{(q_A+q_B)^2} \,. \tag{13}$$
Only a single diagram contributes to this process, and the Feynman rules for the muon are exactly the same as for the electron.

2. Compute the unpolarized amplitude squared, which is obtained by summing over the final state spins r_1, r_2 and averaging over the initial spins s_A, s_B:
$$\frac{1}{4}\sum_{s_A,s_B,r_1,r_2}|M|^2 = \frac{1}{4}\sum_{s_A,s_B,r_1,r_2} M^\dagger M \,. \tag{14}$$
For the conjugate Dirac structure recall the relations about γ matrices. Also, the sum over spins can be simplified using the completeness relations.

3. Show that the result reads
$$\frac{1}{4}\sum_{s_A,s_B,r_1,r_2}|M|^2 = \frac{8e^4}{(q_A+q_B)^4}\left(q_A p_2\, q_B p_1 + q_A p_1\, q_B p_2\right) \,. \tag{15}$$
We are neglecting the mass in this exercise.

4. The so-called Mandelstam variables are defined as $s = (q_A+q_B)^2$, $t = (q_A-p_1)^2$, $u = (q_A-p_2)^2$, where q_A, q_B are in-coming four-momenta, p_1, p_2 are out-going four-momenta, and $q_A + q_B = p_1 + p_2$ because of energy-momentum conservation. Show that the Mandelstam variables obey
$$s+t+u = q_A^2 + q_B^2 + p_1^2 + p_2^2 \,. \tag{16}$$

5. Show that in terms of Mandelstam variables, the result in eq. (12) can be written as
$$\frac{1}{4}\sum_{s_A,s_B,r_1,r_2}|M|^2 = 2e^4\frac{t^2+u^2}{s^2} \,. \tag{17}$$

6. It is interesting to consider the slightly more complicated process of $e^+e^- \to e^+e^-$ in QED, which receives contributions from two diagrams. Upon squaring the amplitude, including the interference term, show that in this case the matrix element reads:
$$\frac{1}{4}\sum_{s_A,s_B,r_1,r_2}|\mathcal{M}^{(e^+e^-\to\mu^+\mu^-)}|^2 = 4e^4\frac{(s^2+st+t^2)^2}{s^2t^2} \,. \tag{18}$$
Verify that both results above satisfy the symmetry expected from interchanging appropriate combinations of initial- and final-state momenta (i.e. crossing symmetry).

   Such computations are key components of many High-Energy Physics results, and it is both error-prone and tedious to do them by hand. We will discuss their automation in future exercises, but for now I encourage you to visit `https://feyncalc.github.io/examples` showcasing the use of `FeynCalc`, a `Mathematica` package for symbolic computations in High-Energy Physics.