10 Cross-sections and decay rate

The previous exercises focused on the computation of the scattering matrix M. We will now see how to use M to compute the probability P that two particles A and B scatter into a given set of final-state particles. This probability depends on both the particles and their interactions as well as the density of the incoming particles and their relative velocity. The cross-section σ is thus defined by dividing out these factors:

$$\frac{dP}{dt\, d^3\vec{x}} = |\vec{v}_A - \vec{v}_B|\rho_A(x)\rho_B(x)\sigma \ , \tag{1}$$

where it is usually assumed that the incoming particles both fly along the z-axis and $v_i = p_i^z/E_i$. Such cross-sections are essential in High-Energy Physics experiments as they constitutes the final theoretical predictions that can be compared against the data gathered in the detectors. These cross-sections indirectly encode all information about the fundamental interactions and properties of the underlying Lagrangian which can thus be extracted from the data. The $2 \to n$ cross section is given by

$$\mathrm{d}\sigma = \frac{1}{F}\prod_{i=1}^{n}\frac{\mathrm{d}^3\vec{p}_i}{(2\pi)^3 2E_i}\left|\mathcal{M}\left(q_A, q_B \to \{p_f\}\right)\right|^2 (2\pi)^4\delta^{(4)}\left(q_A + q_B - \sum_{i=1}^{n} p_i\right) \ , \tag{2}$$

where F is called the flux factor.

1. If we place ourselves in the rest frame of particle B, we can replace $F = 2E_A 2E_B|\vec{v}_A - \vec{v}_B|$ by the unambiguous $F = 4E_A m_B|\vec{v}_A|$. Show that this can be expressed in a Lorentz-invariant form as $F = 4\sqrt{(q_A\cdot q_B)^2 - m_A^2 m_B^2}$.

To obtain eq. (2), one computes the probability P for scattering normalized Gaussian wave packets of the form

$$|\phi(\vec{p}, L)\rangle \equiv \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}\phi(\vec{k}, \vec{p}, L)|\psi(\vec{k})\rangle \tag{3}$$

with

$$\phi(\vec{k}, \vec{p}, L) = \mathcal{N} e^{-(\vec{p}-\vec{k})^2 L^2} \tag{4}$$

and $|\vec{p}| \gg L^{-1}$. The normalisation factor is chosen such that $\langle\phi(\vec{p}, L)|\phi(\vec{p}, L)\rangle = 1$. The particle density $\rho(x) = |\phi(x)|^2$ is obtained from the Fourier transform.

$$\phi(x) = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\phi(\vec{k}, \vec{p}, L)e^{-ik\cdot x}. \tag{5}$$

For the derivation one uses the fact that the wave packets are sharply peaked at $\vec{k} \approx \vec{p}$, so that one can replace the momenta $\vec{k}$ in the amplitudes by $\vec{p}$, the typical momentum of the wave packet, up to corrections of order $|\vec{p}| \gg 1/L$. In practical computations, cross-sections are oten computed considering convolutions of the *partonic* cross-section with particular beam profiles accounting for the energy spread of the colliding particles.

Apart from cross-sections, we are also interested in decay rates. In the limit where the decay width Γ is much smaller than the particle mass M, and we are in the rest frame of the decaying particle, its decay into a particular final state can be computed as

$$\mathrm{d}\Gamma = \frac{1}{2M}\prod_{i=1}^{n}\frac{\mathrm{d}^3\vec{p}_i}{(2\pi)^3 2E_i}\left|\mathcal{M}\left(q \to \{p_f\}\right)\right|^2 (2\pi)^4\delta^{(4)}\left(q - \sum_{i=1}^{n} p_i\right) \ . \tag{6}$$

The total decay rate Γ_{tot} of an unstable particle is the sum of the rates into all possible decay channels and the lifetime is $\tau = 1/\Gamma_{tot}$.

11 Phase space integrals

Both the scattering amplitude and the particle decay require that one evaluates phase-space integrals, i.e. integrals of the form

$$\mathrm{d}\Phi_n\left(q \to \{p_f\}\right) = \prod_{i=1}^{n} \frac{\mathrm{d}^3\vec{p}_i}{(2\pi)^3 2E_i} (2\pi)^4 \delta^{(4)}\left(q - \sum_{i=1}^{n} p_i\right) . \tag{7}$$

To compute these, one eliminates some integrals using momentum conservation. The remaining ones are parameterized in terms of suitable variables, for example energies and angles.

1. Show that after integrating over selected variables in order to remove δ-functions, the two-body phase space takes the form

$$\mathrm{d}\Phi_2\left(q \to p_1, p_2\right) \to \frac{\mathrm{d}\cos\theta \mathrm{d}\phi}{(2\pi)^2} \frac{|\vec{p}_1|}{4\sqrt{s}}, \quad s = q^2 . \tag{8}$$

   The phase-space is Lorentz invariant, but the angles and the three momentum refer to the rest frame of the decaying particle. The angles θ and ϕ are relative to some fixed axis in the rest frame of q. For a $2 \to 2$ scattering process $q = q_A + q_B$ and the rest frame of q is the center-of-mass frame and θ is chosen as the angle between $\vec{q}_A$ and $\vec{p}_1$.

2. Show that the three momentum is given by

$$|\vec{p}_1| = |\vec{p}_2| = \frac{1}{2\sqrt{s}} \sqrt{\lambda\left(s, p_1^2, p_2^2\right)} , \tag{9}$$

   where $\lambda(a,b,c) = (a-b-c)^2 - 4bc$. For the scattering $e^+e^- \to \mu^+\mu^-$, we have $p_1^2 = p_2^2 = m_\mu^2$, but derive the formula for the general case of unequal masses.

3. Using this result, show that the $2 \to 2$ scattering cross section (2), evaluated in the center-of-mass frame, simplifies to

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_{\text{c.m.}}} = \frac{1}{F} \frac{|\vec{p}_1|}{16\pi^2 E_{\text{c.m.}}} \left|\mathcal{M}\left(q_A, q_B \to p_1, p_2\right)\right|^2 , \tag{10}$$

   where $\Omega_{\text{c.m.}}$ is the solid angle of particle 1 and $E_{\text{c.m.}} = E_A + E_B$ in this frame. Since the cross section does not depend on the azimuthal angle, we can write $d\Omega_{\text{c.m.}} = 2\pi \sin\theta_{\text{c.m.}}\, d\theta_{\text{c.m.}}$, where $\theta_{\text{c.m.}}$ is the scattering angle in the center of mass frame.

4. Compute the total cross section for $2 \to 2$ scattering in ϕ^4 theory in the center-of-mass frame at a given center-of-mass energy.

5. Finally, evaluate the cross section for $e^+e^- \to \mu^+\mu^-$ following from

$$\frac{1}{4} \sum_{s_A, s_B, r_1, r_2} |M|^2 = 2e^4 \frac{t^2 + u^2}{s^2} , \tag{11}$$

   derived last time. Work in the center-of-mass frame in the high-energy limit, where one can neglect the electron and muon masses. We choose to parameterize the momenta as

$$\begin{aligned} q_A &= E(1,0,0,1), & q_B &= E(1,0,0,-1) \\ p_1 &= E(1,\sin\theta,0,\cos\theta), & p_2 &= E(1,-\sin\theta,0,-\cos\theta) \end{aligned} . \tag{12}$$

(a) Show that the differential muon production cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha_{\mathrm{em}}^2}{4s}\left(1+\cos^2\theta\right), \quad \alpha_{\mathrm{em}} \equiv \frac{e^2}{4\pi}, \tag{13}$$

and sketch the physical meaning of this result.

(b) Show that the total cross section reads

$$\sigma = \frac{4\pi\alpha_{\mathrm{em}}^2}{3s}. \tag{14}$$

6. Identify what prevents one from computing the similar inclusive cross-section for the Bhabha scattering process $e^+e^- \to e^+e^-$, whose matrix element you also computed last time. Explain how realistic fiducial volume cuts relevant for actual collider experiments alleviate this problem.