## 11 Loop calculations: analytic approach

Consider a theory of one real scalar field $\phi$ and two chiral fermions $\psi_L$ and $\psi_R$ with a $U(1)$ gauge symmetry with the coupling strength $e$ and a global $Z_2$ symmetry. Field charges under these symmetries are listed in the table below.

Table 1: Field charge assignment.

| | $U(1)$ | $Z_2$ |
|---|---|---|
| $\phi$ | 0 | -1 |
| $\psi_L$ | $Q$ | +1 |
| $\psi_R$ | $Q$ | -1 |

1. Write down the most general Lagrangian up to dimension 4. Canonically normalize the kinetic terms and keep the coefficients of the remaining operators general.

2. What fields are allowed to have mass and what fields do not. Why not?

   You should arrive at the following Lagrangian (we will adopt the following coefficients for the rest of the problem sheet)

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4 \\ &+ \bar{\psi}_L\, i\not{D}\psi_L + \bar{\psi}_R\, i\not{D}\psi_R - y\phi\bar{\psi}_L\psi_R - y\phi\bar{\psi}_R\psi_L\,.\end{aligned} \tag{1}$$

   Consider all the parameters $\mu$, $\lambda$ and $y$ to be real.

3. Find the minimum $v$ of the scalar potential ($\mu^2 > 0$ since $\mu \in \mathbb{R}$).

4. The scalar field spontaneously breaks the symmetry by acquiring VEV. Consider the positive value of VEV and expand the scalar field around its minimum $\phi = v+S$. Rewrite the Lagrangian in terms of the new scalar field $S$ instead of $\phi$. You can drop the constant term.

5. What fields acquire a mass after the spontaneous symmetry breaking? Express their masses in terms of the parameters $\mu$, $y$ and $v$.

6. Rewrite the Lagrangian once more with all the parameters written in terms of the particle masses and $v$.

   The resulting Lagrangian should look as follows

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu S\partial^\mu S - \frac{m_S^2}{2}S^2 - \frac{m_S^2}{2v}S^3 - \frac{m_S^2}{8v^2}S^4 \\ &+ \bar{\psi}\, i\not{D}\psi - m_\psi\bar{\psi}\psi - \frac{m_\psi}{v}S\bar{\psi}\psi\,,\end{aligned} \tag{2}$$

   where $\psi = \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}$ such that $\psi_L = P_L\psi$ and $\psi_R = P_R\psi$ with $P_{L/R} = \frac{1}{2}(1 \mp \gamma^5)$.

7. What are the momentum space Feynman rules for the propagators of the theory?

8. What are the momentum space Feynman rules for the vertices of the theory?

9. Assume the mass hierarchy $m_\gamma < m_S < m_\psi$. Reason that the dominant decay channel for the scalar field is $S \to 2\gamma$. Draw all the Feynman diagrams for this process. (Note: $\gamma$ is the $U(1)$ gauge boson.)

10. Calculate the scattering amplitude for the scalar decay $\mathcal{M}_{\lambda,\lambda'}(S \to 2\gamma)$, where $\lambda$ and $\lambda'$ label polarizations of the outgoing photons. You should find

$$\mathcal{M}_{\lambda,\lambda'} = \frac{e^2 Q^2}{2\pi^2 v} \epsilon^*_\mu(p,\lambda)\epsilon^*_\nu(q,\lambda') \left[p^\nu q^\mu - g^{\mu\nu}\frac{m_S^2}{2}\right] \mathcal{I}\left(\frac{m_S^2}{m_\psi^2}\right) , \tag{3}$$

where

$$\mathcal{I}(a) = \int_0^1 dx \int_0^{1-x} dy \frac{1-4xy}{1-axy} . \tag{4}$$

11. Calculate the scattering amplitude squared and sum over final state polarizations $\sum_{\lambda,\lambda'} \left|\mathcal{M}_{\lambda,\lambda'}\right|^2$.

12. Calculate the scalar partial decay width $\Gamma(S \to 2\gamma)$. You should find the result

$$\Gamma(S \to 2\gamma) = \frac{e^4 Q^4 m_S^3}{2^8 \pi^5 v^2} \left|\mathcal{I}(a)\right|^2 \tag{5}$$

13. Find the explicit expression for $\mathcal{I}(a)$ (using Mathematica or other software is allowed). What is the behaviour of this function in the limit $m_\psi \to \infty$? What is the physical interpretation of this limit when applied to $\Gamma(S \to 2\gamma)$?
*Hint: it is important to carefully specify how the limit $m_\psi \to \infty$ can be taken in terms of the free parameters of the unbroken lagrangian.*

14. Consider now the case where $2m_\psi < m_S$. What is expected to be the dominant channel for the scalar field $S$ in this case? Compute this signficantly simpler tree-level decay rate.

15. The decay rate $\Gamma(S \to 2\gamma)$ still contributes in this regime, but its computation requires additional care. Explain what is the problem in the naive computation of the integral of eq. (4) and the resulting ambiguity in the evaluation of the analytic expression you found for $\mathcal{I}(a)$, which is now a multi-valued function. How can each of these two expressions be properly regularized?
*Hint: Consider how Feynman's causal prescription $+i\epsilon$ of the denominator of the loop propagators can be thought of as the spinor mass acquiring an infintesimal imaginary part.*

You may find the following identities useful.

Feynman parameters:

$$\frac{1}{ABC} = \int_0^1 dxdydz \frac{2\delta(x+y+z-1)}{(xA+yB+yC)^3} \tag{6}$$

Trace of gamma matrices:

$$\mathrm{Tr}[\text{odd } \# \text{ of } \gamma] = 0 \tag{7}$$

$$\mathrm{Tr}[\gamma^\mu \gamma^\nu] = 4g^{\mu\nu} \tag{8}$$

$$\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] = 4[g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}] \tag{9}$$

Photon transversality:

$$\varepsilon_\mu(p) p^\mu = 0 \tag{10}$$

Loop integrals:

$$\int \frac{d^d l}{(2\pi)^d}\, l^\mu l^\nu = \frac{g^{\mu\nu}}{d} \int \frac{d^d l}{(2\pi)^d}\, l^2 \tag{11}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{1}{[l^2-\Delta]^2} = \frac{i}{(4\pi)^{d/2}} \frac{1}{\Delta^{2-d/2}} \Gamma\left(\frac{4-d}{2}\right) \tag{12}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^2}{[l^2-\Delta]^2} = -\frac{d}{2} \frac{i}{(4\pi)^{d/2}} \frac{1}{\Delta^{1-d/2}} \Gamma\left(\frac{2-d}{2}\right) \tag{13}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{1}{[l^2-\Delta]^3} = \frac{-i}{2(4\pi)^{d/2}} \frac{1}{\Delta^{3-d/2}} \Gamma\left(\frac{6-d}{2}\right) \tag{14}$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l^2}{[l^2-\Delta]^3} = \frac{d}{4} \frac{i}{(4\pi)^{d/2}} \frac{1}{\Delta^{2-d/2}} \Gamma\left(\frac{4-d}{2}\right) \tag{15}$$

Differential decay rate and cross section:

$$d\Gamma = \frac{1}{2M} |\mathcal{M}|^2 \, d\Phi_2 \,, \tag{16}$$

$$\sigma = \frac{1}{S} \int d\sigma \,, \tag{17}$$

where $S = 1$ for decay into two distinct particles while $S = 2$ for decay into two identical particles and $d\phi_2$ is the two-body phase space we evaluated last week.