Series 1

As a prelude, let us first show the following two useful results:

$$I_1(J,a) \equiv \int_{-\infty}^{+\infty} \mathrm{d}x\, e^{-\frac{1}{2}ax^2+Jx} = \sqrt{\frac{2\pi}{a}} \cdot e^{\frac{J^2}{2a}}, \tag{1}$$

where J and a are real numbers, and

$$I_2(J,A) \equiv \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \mathrm{d}x_1 \ldots \mathrm{d}x_N\, e^{-\frac{1}{2}x^\top Ax+J^\top x} = \frac{(2\pi)^{\frac{N}{2}}}{\sqrt{\det(A)}} \cdot e^{\frac{1}{2}J^\top A^{-1}J}, \tag{2}$$

where J is an N-dimensional vector and A is a real symmetric $N \times N$ matrix.

We first solve $I_1(J,a)$, and then use it to compute $I_2(J,A)$. For this, we need to complete the square in the exponent in order to be able to perform a suitable change of variables. We write

$$-\frac{1}{2}ax^2 + Jx = -\frac{1}{2}a\left(x - \frac{J}{a}\right)^2 + \frac{J^2}{2a} = -\frac{1}{2}ay^2 + \frac{J^2}{2a}, \tag{3}$$

where we defined $y = x - \frac{J}{a}$. Therefore,

$$I_1(J,a) = \int_{-\infty}^{+\infty} \mathrm{d}y\, e^{-\frac{1}{2}ay^2+\frac{J^2}{2a}} = e^{\frac{J^2}{2a}} \left[\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathrm{d}y\mathrm{d}z\, e^{-\frac{1}{2}a(y^2+z^2)}\right]^{\frac{1}{2}}. \tag{4}$$

The last step might seem useless: we squared the integral, and then took the square root. However, it is important to note that we are now in position to move to polar coordinates:

$$\begin{aligned} I_1(J,a) &= e^{\frac{J^2}{2a}} \left[2\pi \int_0^{+\infty} \mathrm{d}r\, re^{-\frac{1}{2}ar^2}\right]^{\frac{1}{2}} \\ &= e^{\frac{J^2}{2a}} \left[-\frac{2\pi}{a}e^{-\frac{1}{2}ar^2}\Big|_0^{+\infty}\right]^{\frac{1}{2}} \\ &= e^{\frac{J^2}{2a}} \cdot \left(\frac{2\pi}{a}\right)^{\frac{1}{2}}, \end{aligned} \tag{5}$$

as was to be shown. Next, we know that there exists a real orthogonal $N \times N$ matrix O (i.e. $O^\top = O^{-1}$) such that $A = O^T DO$, where $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ is a diagonal matrix whose entries are the eigenvalues λ_i of A. Let also $y = Ox$ and $H = OJ$. Then,

$$-\frac{1}{2}x^\top Ax + J^\top x = -\frac{1}{2}y^\top Dy + H^\top y = \sum_{i=1}^{N}\left[-\frac{1}{2}\lambda_i y_i^2 + H_i y_i\right], \tag{6}$$

so that the integration variables y_i don't mix with each other. Hence, they are dummy variables and we relabel all of them as y, i.e.

$$\begin{aligned} I_2(J,A) &= \prod_{i=1}^{N} \int_{-\infty}^{+\infty} \mathrm{d}y\, e^{-\frac{1}{2}\lambda_i y^2 + H_i y} \\ &= \prod_{i=1}^{N} e^{\frac{H_i^2}{2\lambda_i}} \cdot \left(\frac{2\pi}{\lambda_i}\right)^{\frac{1}{2}} \\ &= \left(\frac{(2\pi)^N}{\prod_{i=1}^{N}\lambda_i}\right)^{\frac{1}{2}} \cdot e^{\frac{1}{2}\sum_{i=1}^{N}\frac{H_i^2}{\lambda_i}}, \end{aligned} \tag{7}$$

where we used the result for $I_1(J,a)$ in the second line. Now, note that $\prod_{i=1}^{N}\lambda_i = \det(A)$ and $D^{-1} = \mathrm{diag}(\lambda_1^{-1},\ldots,\lambda_N^{-1})$, which yields

$$\sum_{i=1}^{N}\frac{H_i^2}{\lambda_i} = H^\top D^{-1} H = (OJ)^\top D^{-1}(OJ) = J^\top A^{-1} J \tag{8}$$

because $A^{-1} = O^\top D^{-1} O$, which proves Eq. (2).

I. Complex Gaussian integrals

Calculate the integral

$$\int (\mathrm{d}\vec{z}^*)(\mathrm{d}\vec{z}) e^{-(\vec{z}^*, A\vec{z})}, \tag{9}$$

where $\vec{z} = \frac{1}{\sqrt{2}}(\vec{x} + i\vec{y})$, $(\mathrm{d}\vec{z}^*)(\mathrm{d}\vec{z}) = (\mathrm{d}\vec{x})(\mathrm{d}\vec{y})$ *and* A *is a real, symmetric* $N \times N$ *matrix.*

In standard matrix notation, the exponent reads

$$\begin{aligned} -\bar{z}^\top A z &= -(x - iy)^\top A(x + iy) \\ &= -x^\top A x - y^\top A y - i x^\top A y + i y^\top A x \\ &= -x^\top A x - y^\top A y, \end{aligned} \tag{10}$$

where the last two terms of the second line cancel each other because A is symmetric. Hence, the components of x and y do not mix, and this integral is simply the square of $I_2(0,A)$, namely

$$\int (\mathrm{d}\vec{z}^*)(\mathrm{d}\vec{z}) e^{-(\vec{z}^*, A\vec{z})} = \frac{(2\pi)^N}{\det(A)}. \tag{11}$$

Note that, by convention, the factor of $(2\pi)^N$ could have been absorbed in the measure, as proposed in Eq. (0.4a) of the notes.

II. Wick contractions

1. *Calculate the expression*

$$\langle x^{2n} \rangle = \frac{\int_{-\infty}^{+\infty} \mathrm{d}x\, e^{-\frac{1}{2}ax^2} x^{2n}}{\int_{-\infty}^{+\infty} \mathrm{d}x\, e^{-\frac{1}{2}ax^2}}. \tag{12}$$

Powers of x can be generated in the integral upon differentiating $I_1(J,a)$ with respect to either J or a and setting $J=0$. More precisely,

$$\langle x^{2n}\rangle = \frac{(-2)^n}{I_1(0,a)}\frac{\partial^n I_1(0,a)}{\partial a^n} = \frac{1}{I_1(0,a)}\frac{\partial^{2n} I_1(J,a)}{\partial J^{2n}}\bigg|_{J=0}. \tag{13}$$

Looking at Eq. (5), one can see that differentiation with respect to a is less cumbersome. Indeed, we have

$$\frac{\partial^n I_1(0,a)}{\partial a^n} = \frac{\partial^n}{\partial a^n}\left[\left(\frac{2\pi}{a}\right)^{\frac{1}{2}}\right] = \left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\dots\cdot\left(-\frac{2n-1}{2}\right)\frac{\sqrt{2\pi}}{a^{\frac{2n+1}{2}}}. \tag{14}$$

Plugging this back in Eq. (13), we find that

$$\langle x^{2n}\rangle = \frac{(2n-1)!!}{a^n}, \tag{15}$$

where $(2n-1)!! \equiv (2n-1)\cdot(2n-3)\cdot\dots\cdot 1$. While this is the final answer, it may be instructive to find this result upon taking derivatives with respect to J instead. It is slightly more complex, though, because successive derivatives generate more and more terms to deal with. Indeed, let $u(J) \equiv \frac{J^2}{2a}$ and let the derivative with respect to J be denoted by a prime. We have

$$e^u \to u'e^u \to [u''+(u')^2]e^u \to [u'''+3u'u''+(u')^3]e^u \to [u''''+4u'''u'+3(u'')^2+6u''(u')^2+(u')^4]e^u \to \text{etc.}, \tag{16}$$

where each arrow indicates an additional derivative. Generally speaking, we'll end up with all possible combinations of terms that contain $2n$ primes. It is important to note, however, that $u(J)$ is quadratic in J, so that $u^{(p)}=0$ when $p>2$. Moreover, remember than we need to set $J=0$ in the end, which means that any term containing u' will vanish. The only piece that survives is the $(u'')^n$-term, and all we need to do is to compute the coefficient that comes with it. This is therefore a combinatorics problem.

Indeed, one can see that the total number of contributions to the $(u'')^n$-term is given by the number of *Wick contractions*, which is the number of ways to distribute $2n$ distinct balls in n unordered boxes that can contain only 2 balls, where the order inside the box doesn't matter. For example, if $n=3$ (i.e. 6 balls and 3 boxes), there are $6! = 720$ different ways to fill in the six available spots in the boxes. Among these, some are regarded as equivalent because we do not care about the order. In particular, the three boxes can be arranged in 3! equivalent ways, and the balls inside each box can be arranged in two equivalent configurations. In the end, we thus have $\frac{6!}{3!\cdot 2^3} = 15$ Wick contractions. More generally, this leads to

$$\frac{(2n)!}{n!\cdot 2^n} = (2n-1)!! \tag{17}$$

possibilities, which can be visualized as follows. Write $\langle x^{2n}\rangle$ as $\langle x\cdot x\cdot\dots\cdot x\rangle$ and link any x to any other x, and repeat until you formed n-pairs:

$$\langle x\cdot x\cdot x\cdot\dots\cdot x\rangle \tag{18}$$

Moreover, since $(u'')^n = \frac{1}{a^n}$, we recover Eq. (15).

2. *Calculate the expression*

$$\langle x_i x_j \rangle = \frac{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \mathrm{d}x_1 \ldots \mathrm{d}x_N \, e^{-\frac{1}{2}x\cdot A\cdot x} x_i x_j}{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \mathrm{d}x_1 \ldots \mathrm{d}x_N \, e^{-\frac{1}{2}x\cdot A\cdot x}}. \tag{19}$$

This is very similar to what we've just done. We have

$$\langle x_i x_j \rangle = \frac{-2}{I_2(0,A)} \frac{\partial I_2(0,A)}{\partial A_{ij}} = \frac{1}{I_2(0,A)} \frac{\partial^2 I_2(J,A)}{\partial J_i \partial J_j}\bigg|_{J=0}. \tag{20}$$

Differentiation with respect to A_{ij} implies some non-trivial results of linear algebra, in particular

$$\frac{\partial \det(A)}{\partial A_{ij}} = \det(A)(A^{-1})_{ij}. \tag{21}$$

Using Eq. (2), it is then easy to see that

$$\frac{\partial I_2(0,A)}{\partial A_{ij}} = \frac{(2\pi)^{\frac{N}{2}}}{\sqrt{\det(A)}} \cdot \frac{(A^{-1})_{ij}}{-2} \tag{22}$$

which implies

$$\langle x_i x_j \rangle = (A^{-1})_{ij}. \tag{23}$$

Let us see what happens if we take the derivative with respect to J instead. We have

$$\begin{aligned} \frac{\partial}{\partial J_i} e^{\frac{1}{2}J^\top A^{-1} J} &= e^{\frac{1}{2}J^\top A^{-1} J} \cdot \frac{1}{2} \sum_{m=1}^{N} [(A^{-1})_{im} J_m + J_m (A^{-1})_{mi}] \\ &= e^{\frac{1}{2}J^\top A^{-1} J} \cdot \sum_{m=1}^{N} (A^{-1})_{im} J_m, \end{aligned} \tag{24}$$

and therefore

$$\frac{\partial^2}{\partial J_i \partial J_j} e^{\frac{1}{2}J^\top A^{-1} J} = e^{\frac{1}{2}J^\top A^{-1} J} \cdot \left[\sum_{m,n=1}^{N} (A^{-1})_{im} J_m (A^{-1})_{jn} J_n + (A^{-1})_{ij} \right]. \tag{25}$$

Setting $J = 0$, we get $\langle x_i x_j \rangle = (A^{-1})_{ij}$ as before. Note that differentiation with respect to J leads to the same issue as before, namely we're getting more and more terms.

3. *Calculate the expression*

$$\langle x_i x_j x_k \rangle = \frac{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \mathrm{d}x_1 \ldots \mathrm{d}x_N \, e^{-\frac{1}{2}x\cdot A\cdot x} x_i x_j x_k}{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \mathrm{d}x_1 \ldots \mathrm{d}x_N \, e^{-\frac{1}{2}x\cdot A\cdot x}}. \tag{26}$$

Here, we note that if we go back to the eigenbasis that diagonalizes A, the exponent is an even function of y, while $x_i x_j x_k$ becomes an odd function of the y's. Since the domain of integration is symmetric, one concludes that $\langle x_i x_j x_k \rangle = 0$. This is also evident from Eq. (29) below, with $J = 0$.

4. *Calculate the expression*

$$\langle x_i x_j x_k x_l \rangle = \frac{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \mathrm{d}x_1 \ldots \mathrm{d}x_N \, e^{-\frac{1}{2} x \cdot A \cdot x} x_i x_j x_k x_l}{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \mathrm{d}x_1 \ldots \mathrm{d}x_N \, e^{-\frac{1}{2} x \cdot A \cdot x}}. \tag{27}$$

Again, one can choose to differentiate with respect to A or J. Let us differentiate with respect to J:

$$\langle x_i x_j x_k x_l \rangle = \frac{1}{I_2(0,A)} \frac{\partial^4 I_2(J,A)}{\partial J_i \partial J_j \partial J_k \partial J_l} \bigg|_{J=0}. \tag{28}$$

Starting from Eq. (25), we obtain (repeated indices are now summed over implicitly)

$$\begin{aligned} \frac{\partial^3}{\partial J_i \partial J_j \partial J_k} e^{\frac{1}{2} J^\top A^{-1} J} = e^{\frac{1}{2} J^\top A^{-1} J} \Big[&(A^{-1})_{im} J_m (A^{-1})_{jn} J_n (A^{-1})_{kp} J_p \\ &+ (A^{-1})_{ij} (A^{-1})_{kp} J_p \\ &+ (A^{-1})_{ik} (A^{-1})_{jn} J_n \\ &+ (A^{-1})_{im} J_m (A^{-1})_{jk} \Big]. \end{aligned} \tag{29}$$

While taking the derivative of this expression with respect to J_l, it is useful to bear in mind that all components of J will then be set to zero, as indicated in Eq. (28). In turn, this means that only the last three lines of the above expression have non-vanishing contributions, namely:

$$\frac{1}{I_2(J,A)} \frac{\partial^4 I_2(J,A)}{\partial J_i \partial J_j \partial J_k \partial J_l} \bigg|_{J=0} = (A^{-1})_{ij} (A^{-1})_{kl} + (A^{-1})_{ik} (A^{-1})_{jl} + (A^{-1})_{il} (A^{-1})_{jk}. \tag{30}$$

5. *What is the expression for a general number of insertions, $\langle x_i x_j \ldots x_k x_l \rangle$?*

For any odd number of insertions, the same argument as before holds, namely: the result vanishes. Let us therefore assume that the number of insertions is even and exploit our previous observations on Wick contractions appearing in $\langle x^{2n} \rangle$ (cf. comment below Eq. (16)). In this particular example, all x's were the same and therefore, one merely needed to *count* the total number of Wick contractions. However, when the x's correspond to different components of the vector x, each Wick contraction corresponds to a different combination of $(A^{-1})_{ab}(A^{-1})_{cd}$, as can be seen from the case of $\langle x_i x_j x_k x_l \rangle$, Eq. (30). Therefore,

$$\langle x_i x_j \ldots x_k x_l \rangle = \sum_{\text{Wick}} (A^{-1})_{ab} \ldots (A^{-1})_{cd}, \tag{31}$$

where the sum runs over all possible Wick contractions, corresponding to the pairs $(ab), \ldots, (cd)$ where $a, b, c, d,$ etc. $\in \{i, j, \ldots, k, l\}$.