Series 2

I. Propagators of the free real scalar field

1. *Calculate the causal retarded Green's function via residues:*

$$D_{ret}(x-y) = \int \frac{\mathrm{d}\vec{k}}{(2\pi)^3}\, e^{i(\vec{x}-\vec{y})\vec{k}} \times \int \frac{\mathrm{d}k_0}{2\pi} \frac{e^{-i(x^0-y^0)k_0}}{(k_0-\omega_k+i\epsilon)(k_0+\omega_k+i\epsilon)}, \tag{1}$$

treating $t > 0$ and $t < 0$ separately. What do we see?

Let us call the integrand $f(k_0) \equiv \frac{e^{-itk_0}}{(k_0-\omega_k+i\epsilon)(k_0+\omega_k+i\epsilon)}$. It has two poles of order 1, when $k_0 = \pm\omega_k - i\epsilon$. Both of these poles are located in the lower half of the complex plane. In order to evaluate the integral over k_0 over the real numbers, we will close the contour in order to take advantage of the residue theorem.

Roughly speaking, it states the following. Let U be an open (simply connected) subset of the complex plane and let f be a holomorphic function on U except at some points $\{z_1, \dots z_k\}$ (= singularities). Then, if Γ is a positively oriented simple closed curve in U that does not meet any of the z_k's, the integral of f over this curve is given by

$$\oint_\Gamma \mathrm{d}z\, f(z) = 2\pi i \sum \mathrm{Res}(f, z_j) \tag{2}$$

where $\mathrm{Res}(f, z_j)$ is the residue of f around $z = z_j$ and the sum runs over all singularities enclosed by Γ. In particular, if z_j is a pole of order n, then

$$\mathrm{Res}(f, z_j) = \frac{1}{(n-1)!} \lim_{z\to z_j} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \left[f(z)\cdot (z-z_j)^n \right]. \tag{3}$$

Let us therefore apply this to the function $f(k_0)$, assuming for now that $t \equiv x^0 - y^0 > 0$. Closing the contour can be done in a smart and convenient way. Indeed, since there's an exponential in the integrand, we can close the contour along an infinitely large semi-circle in the part of the plane where the exponential is damped. The integral along this semi-circle thus vanishes, leaving only the integral along the real axis. This occurs when the imaginary part of k_0 is negative, i.e. the lower half-plane, which means that both poles are enclosed by this contour. We therefore have

$$\begin{aligned} \int \frac{\mathrm{d}k_0}{2\pi} f(k_0) &= \frac{-2\pi i}{2\pi} \left[\mathrm{Res}(f, -\omega - i\epsilon) + \mathrm{Res}(f, \omega - i\epsilon) \right] \\ &= -i \left[\frac{e^{it\omega_k}}{-2\omega_k} + \frac{e^{-it\omega_k}}{2\omega_k} \right] \\ &= \frac{ie^{it\omega_k}}{2\omega_k} - \frac{ie^{-it\omega_k}}{2\omega_k} \\ &= -\frac{\sin(t\omega_k)}{\omega_k} \end{aligned} \tag{4}$$

Note that an extra minus sign is added because the contour was not positively oriented, and ϵ was sent to 0. However, had we assumed that $t < 0$, the contour would not have enclosed any of the poles, and the integral would have been equal to 0. We can therefore write

$$D_{ret}(x-y) = -\theta(x^0 - y^0) \int \frac{\mathrm{d}\vec{k}}{(2\pi)^3 \omega_k} e^{i(\vec{x}-\vec{y})\vec{k}} \sin(\omega_k(x^0 - y^0)) \tag{5}$$

where $\theta(t) = 0$ if $t < 0$ and 1 if $t > 0$ is the Heaviside function. The retarded Green's function only describes propagation in the future (corresponding to $x^0 > y^0$), hence the name.

2. *Calculate the causal advanced Green's function via residues:*

$$D_{adv}(x-y) = \int \frac{\mathrm{d}\vec{k}}{(2\pi)^3} e^{i(\vec{x}-\vec{y})\vec{k}} \times \int \frac{\mathrm{d}k_0}{2\pi} \frac{e^{-i(x^0-y^0)k_0}}{(k_0 - \omega_k - i\epsilon)(k_0 + \omega_k - i\epsilon)}, \tag{6}$$

treating $t > 0$ and $t < 0$ separately. What do we see?

Let $g(k_0) \equiv \frac{e^{-itk_0}}{(k_0-\omega_k-i\epsilon)(k_0+\omega_k-i\epsilon)}$. This function is very similar to the previous one, except that its poles are now located at $k_0 = \pm\omega_k + i\epsilon$, i.e. in the upper half-plane. But this doesn't change the way we close the contour depending on the sign of $t \equiv x^0 - y^0$, which means we actually have the opposite result: we miss both poles when we close the contour in the lower half-plane (corresponding to $t > 0$), and we pick both of them when $t < 0$ upon closing the contour in the upper half-plane. Therefore, let us assume that $t < 0$ for now. We have

$$\begin{aligned}
\int \frac{\mathrm{d}k_0}{2\pi} g(k_0) &= \frac{2\pi i}{2\pi} \left[\mathrm{Res}(g, -\omega - i\epsilon) + \mathrm{Res}(g, \omega - i\epsilon)\right] \\
&= i \left[\frac{e^{it\omega_k}}{-2\omega_k} + \frac{e^{-it\omega_k}}{2\omega_k} \right] \\
&= \frac{ie^{-it\omega_k}}{2\omega_k} - \frac{ie^{it\omega_k}}{2\omega_k} \\
&= \frac{\sin(t\omega_k)}{\omega_k}
\end{aligned} \tag{7}$$

Here, the contour was positively oriented. Finally, we get

$$D_{ret}(x-y) = \theta(y^0 - x^0) \int \frac{\mathrm{d}\vec{k}}{(2\pi)^3 \omega_k} e^{i(\vec{x}-\vec{y})\vec{k}} \sin(\omega_k(x^0 - y^0)) \tag{8}$$

The advanced Green's function only describes propagation in the past, corresponding to $x^0 < y^0$.