Series 3

I. Propagation over spacelike separation

1. *Verify that the propagator $D(x)$ of the massive scalar (in 4d) vanishes for spacelike separation.*

   Since the propagator is invariant under Lorentz transformations, we can bring any spacelike vector in the form $x \to (0, \vec{x})$. Therefore, we need to integrate the following:
$$D(x) = -i \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{e^{-i\vec{k}\cdot\vec{x}}}{\sqrt{\vec{k}^2 + m^2}}. \tag{1}$$
   In fact, we can further rotate $\vec{x}$ so that it is aligned on the z-axis $\vec{x} \to (0, 0, r)$ where $r = |\vec{x}|$. These transformations can be formally performed upon sending $x^\mu \to \Lambda^\mu{}_\nu x^\nu$ where Λ is a Lorentz transformation with unit determinant, and then changing integration variables. Moreover, using spherical coordinates
$$\vec{k} = k(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi), \tag{2}$$
   where $k \in [0, \infty]$, $\phi \in [0, \pi]$, $\theta \in [0, 2\pi[$ and $\mathrm{d}^3 k = k^2 \sin\phi \mathrm{d}k \mathrm{d}\phi \mathrm{d}\theta$, we have
$$\vec{k} \cdot \vec{x} = kr\cos\phi, \tag{3}$$
   and integration over θ only contributes a factor of 2π. Then,
$$\begin{aligned} D(x) &= \frac{-i}{4\pi^2} \int_0^\infty \frac{\mathrm{d}k\, k^2}{\sqrt{k^2 + m^2}} \int_{-1}^1 \mathrm{d}(\cos\phi) e^{-ikr\cos\phi} \\ &= \frac{-1}{4\pi^2 r} \int_0^\infty \frac{\mathrm{d}k\, k}{\sqrt{k^2 + m^2}} \left(e^{ikr} - e^{-ikr}\right). \end{aligned} \tag{4}$$
   At this point, one could either recognize a modified Bessel function, or extend the integral in the complex plane, taking great care of the branch cuts that the square root involves. The first option gives
$$D(x) = \frac{-i}{2\pi^2 r} \int_0^\infty \mathrm{d}k \frac{k \sin kr}{\sqrt{k^2 + m^2}} = \frac{-im}{2\pi^2 r} K_1(mr) \sim e^{-mr} \tag{5}$$
   where $K_\alpha(z)$ is the modified Bessel function of the second kind, which decays exponentially in our case. But let's have a look at the second option. We want to take advantage of the residue theorem, so we write the propagator as an integral over the real line as follows:
$$D(x) = \frac{-1}{4\pi^2 r} \int_{-\infty}^\infty \mathrm{d}k \frac{k \cdot e^{ikr}}{\sqrt{k^2 + m^2}}. \tag{6}$$
   We close the counter in the upper half-plan because the exponential is suppressed, but we have to go around the branch cut that goes from im to $i\infty$. However, there is no pole enclosed by this contour, so the integral over it

gives 0. Hence, the integral over the real line is equal to twice the integral from im to $i\infty$. Changing variable $k = imu$ where $u \in [1, \infty]$, we get

$$\begin{aligned} D(x) &= \frac{-im}{2\pi^2 r} \int_1^\infty \mathrm{d}u \, \frac{u \cdot e^{-mru}}{\sqrt{u^2-1}} \\ &= \frac{-im}{2\pi^2 r} \int_0^\infty \mathrm{d}t \, \cosh t \cdot e^{-mr\cosh t} \\ &= \frac{-im}{4\pi^2 r} \int_{-\infty}^\infty \mathrm{d}s \, e^{-mr\sqrt{s^2+1}} \end{aligned} \tag{7}$$

where we further performed the changes of variables $u = \cosh t$ ($t \in [0, \infty]$) and $s = \sinh t$ ($s \in [0, \infty]$). If we assume $m \cdot r$ to be sufficiently large, we can conclude thanks to the saddle-point (or steepest-descent) approximation, i.e.

$$\begin{aligned} \int_{-\infty}^\infty \mathrm{d}x \, e^{-f(x)} &= \int_{-\infty}^\infty \mathrm{d}x \, e^{-\left[f(x_0)+\frac{f''(x_0)}{2}(x-x_0)^2+\dots\right]} \\ &\approx e^{-f(x_0)} \sqrt{\frac{2\pi}{f''(x_0)}}, \end{aligned} \tag{8}$$

where x_0 is the minimum of $f(x)$, which gives in our case

$$D(x) \approx -\frac{i}{2} \sqrt{\frac{m}{(2\pi r)^3}} e^{-mr} \tag{9}$$

II. Propagator of the massless free scalar field in 2d

1. Calculate the propagator for the massless free scalar field in 2d by solving the differential equation

$$-\partial_x^2 D(\vec{x}-\vec{y}) = \delta(\vec{x}-\vec{y}). \tag{10}$$

A simple trick to compute this propagator is the following. Let $\vec{r} \equiv \vec{x} - \vec{y}$ and D be a disk of radius $r = |\vec{r}|$ centred at $\vec{y}$ over which we are going to integrate the differential equation:

$$\int_D \mathrm{d}^2x \, \partial_{\vec{x}}^2 D(\vec{x}-\vec{y}) = -\int_D \mathrm{d}^2x \, \delta(\vec{x}-\vec{y}). \tag{11}$$

In particular, the right-hand side gives -1 because $\vec{y} \in D$. Moreover, it is easy to see that $\partial_{\vec{x}}$ and $\partial_{\vec{x}-\vec{y}}$ act in the same way on functions that only depends on $\vec{x} - \vec{y}$. This implies that the system is invariant under translations and rotations. It thus only depends on r, and we have

$$\partial_{\vec{x}}^2 D(r) = \frac{1}{r^{d-1}} \frac{\partial}{\partial_r} \left(r^{d-1} \frac{\partial}{\partial_r} D(r) \right) \tag{12}$$

where $d = 2$ in our case. Using polar coordinates with $\mathrm{d}^2x = 2\pi\rho\mathrm{d}\rho$ where $\rho \in [0, r]$, we get

$$2\pi r \frac{\partial D(r)}{\partial r} = -1. \tag{13}$$

It is then straightforward to conclude that

$$D(r) = -\frac{1}{2\pi} \ln \frac{r}{r_0} \tag{14}$$

where r_0 is an arbitrary constant length.

III. Potential energy between two static sources

1. *Calculate the integral*

$$E = -\int \frac{\mathrm{d}^3 k}{(2\pi)^2} \frac{e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}}{\vec{k}^2+m^2}. \tag{15}$$

This is essentially the same computation as exercise I, but without the nasty part about branch cuts. As explained before, we can express $\vec{k}$ in spherical coordinates where $\vec{x}_1 - \vec{x}_2 \equiv \vec{r}$ is assumed to be aligned on the z-axis, which yields

$$\begin{aligned} E &= -\frac{1}{4\pi^2} \int_0^\infty \mathrm{d}k \, \frac{k^2}{k^2+m^2} \int_{-1}^{1} \mathrm{d}(\cos\phi) \, e^{-ikr\cos\phi} \\ &= \frac{i}{4\pi^2 r} \int_0^\infty \mathrm{d}k \, \frac{k}{k^2+m^2} \left(e^{ikr} - e^{-ikr}\right) \\ &= \frac{i}{4\pi^2 r} \int_{-\infty}^\infty \mathrm{d}k \, \frac{k \cdot e^{ikr}}{(k+im)(k-im)}. \end{aligned} \tag{16}$$

Now that we have an integral over the real line, we can close the contour. Thanks to the exponential, the semi-circular part of the contour vanishes if it is taken in the upper half-plane, and we thus need to worry about the $+im$ pole. Applying the residue theorem, we simply get

$$E = -\frac{1}{4\pi r} e^{-mr}. \tag{17}$$