Series 4

I. The potential energy in $d = 2+1$ dimensions

1. *Calculate the analog of formula (1.69) in the notes in* $2+1$ *dimensions.*

   The computation of the Yukawa potential in $D = 2$ spatial dimensions is harder than the higher dimensional cases because the Jacobian of the polar coordinates doesn't bring any trigonometric function in the measure[1]:

$$E(r) = -\int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{e^{i\vec{k}\vec{r}}}{\vec{k}^2 + m^2} = -\int_0^\infty \frac{\mathrm{d}k}{(2\pi)^2} \int_0^{2\pi} \mathrm{d}\phi \, \frac{k e^{ikr\sin\phi}}{k^2 + m^2}. \tag{1}$$

   But this was a crucial step in $D = 3$ (or higher), as one could then integrate over ϕ (see Ex. III, Series 3), extend the domain of integration to $\mathbb{R}$ and use complex analysis techniques. Here, none of this is doable as such, and we are going to solve it in a very different manner. In fact, let us even solve the general $(D+1)$-dimensional case, as it won't bring any new layer of complexity. We thus want to solve

$$E(r) = -\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{e^{i\vec{k}\vec{r}}}{\vec{k}^2 + m^2}. \tag{2}$$

   Upon inspecting Eq. (1), one may already recognize some sort of Bessel function. In fact, we shall soon recognize a modified Bessel function of the second kind:

$$K_n(x) = \int_0^\infty \mathrm{d}v \, e^{-x\cosh v} \cosh(nv). \tag{3}$$

   A very nice trick to converge towards such an expression is called the Schwinger parametrization:

$$\frac{1}{\vec{k}^2 + m^2} = \frac{1}{2} \int_0^\infty \mathrm{d}t \, e^{-\frac{t}{2}(k^2 + m^2)}. \tag{4}$$

   It makes it possible to bring the denominator in an exponential form, so as to then be able to play with Gaussian integrals. Indeed, let us remark that

$$i\vec{k}\vec{r} - \frac{t}{2}\left(\vec{k}^2 + m^2\right) = -\frac{t}{2}\left(\vec{k} - \frac{i\vec{r}}{t}\right)^2 - \frac{tm^2}{2} - \frac{\vec{r}^2}{2t}. \tag{5}$$

   It is now easy to perform the following steps:

$$\begin{aligned} E(r) &= -\int \frac{\mathrm{d}^D k}{2(2\pi)^D} \int_0^\infty \mathrm{d}t \, e^{i\vec{k}\vec{r} - \frac{t}{2}\left(\vec{k}^2 + m^2\right)} \\ &= -\int_0^\infty \mathrm{d}t \, e^{-\left(\frac{tm^2}{2} + \frac{\vec{r}^2}{2t}\right)} \int \frac{\mathrm{d}^D k}{2(2\pi)^D} e^{-\frac{t}{2}\left(\vec{k} - \frac{i\vec{r}}{t}\right)^2} \\ &= -\int_0^\infty \frac{\mathrm{d}t}{2(2\pi)^{\frac{D}{2}}} e^{-\frac{mr}{2}\left(\frac{mt}{r} + \frac{r}{mt}\right)} \cdot t^{-\frac{D}{2}}, \end{aligned} \tag{6}$$

[1] Note that, thanks to spherical symmetry, the potential energy can only be a function of the distance $r = |\vec{r}|$ between the two static sources.

where the second integral in the second line is simply a Gaussian integral that gives $(2\pi/t)^{\frac{D}{2}}$. We now make two consecutive changes of variables, namely $u = \frac{r}{mt}$ and $v = \ln u$, which imply

$$\begin{aligned} E(r) &= -\int_0^\infty \frac{\mathrm{d}u}{2(2\pi)^{\frac{D}{2}}} \frac{e^{-\frac{mr}{2}\left(u+\frac{1}{u}\right)}}{u^{\frac{D}{2}-1}} \\ &= -\left(\frac{m}{r}\right)^{\frac{D}{2}-1} \int_{-\infty}^{\infty} \frac{\mathrm{d}v}{2(2\pi)^{\frac{D}{2}}} e^{-mr\cosh v - nv}, \end{aligned} \tag{7}$$

where $n \equiv \frac{D}{2} - 1$. This is almost the standard expression of the modified Bessel function of the second kind $K_n(x)$ with $x = mr$. To make this clear, let us split the domain of integration in two halves $[-\infty, 0]$ and $[0, \infty]$ and change $v \to -v$ in the first one. It yields

$$\begin{aligned} E(r) &= -\left(\frac{m}{r}\right)^{\frac{D}{2}-1} \int_0^\infty \frac{\mathrm{d}v}{(2\pi)^{\frac{D}{2}}} e^{-mr\cosh v} \left[\frac{e^{nv}}{2} + \frac{e^{-nv}}{2}\right] \\ &= -\left(\frac{m}{r}\right)^{\frac{D}{2}-1} \int_0^\infty \frac{\mathrm{d}v}{(2\pi)^{\frac{D}{2}}} e^{-mr\cosh v} \cosh(nv) \end{aligned} \tag{8}$$

and finally,

$$E(r) = -\left(\frac{m}{r}\right)^{\frac{D}{2}-1} \frac{K_n(mr)}{(2\pi)^{\frac{D}{2}}}. \tag{9}$$

For $D = 2$, we simply have

$$E(r) = -\frac{1}{2\pi} K_0(mr). \tag{10}$$

Let us conclude this computation with two remarks. First, let us check that the $D = 3$ case indeed gives the known answer Eq. (1.69) in the notes. Using $K_{1/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x}$, we get

$$E(r) \overset{D=3}{=} -\frac{1}{4\pi r} e^{-mr} \tag{11}$$

as it should! Moreover, for small values of the argument (more precisely, when $|x| \ll \sqrt{n+1}$), we can use the asymptotic behavior $K_n(x) = \frac{\Gamma(n)}{2}\left(\frac{2}{x}\right)^n$ for $n > 0$. This implies that the massless limit $m \to 0$ of Eq. (9) for $D > 2$ is given by

$$E(r) \overset{m=0}{=} -\frac{\Gamma\left(\frac{D}{2} - 1\right)}{4\pi^{\frac{D}{2}}} \frac{1}{r^{D-2}}. \tag{12}$$

II. The massive vector meson

1. *Starting from*

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu + A_\mu J^\mu, \tag{13}$$

arrive via partial integration at the action

$$S = \int \mathrm{d}^4x \left(\frac{1}{2} A_\mu [(\partial^2 + m^2) g^{\mu\nu} - \partial^\mu \partial^\nu] A_\nu + A_\mu J^\mu \right). \tag{14}$$

We only need to focus on the first term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, which we first develop:

$$\begin{aligned} -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} &= -\frac{1}{4}[\partial_\mu A_\nu - \partial_\nu A_\mu][\partial^\mu A^\nu - \partial^\nu A^\mu] \\ &= -\frac{1}{4}[\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu - \partial_\nu A_\mu \partial^\mu A^\nu + \partial_\nu A_\mu \partial^\nu A^\mu]. \end{aligned} \tag{15}$$

Now, because ν and μ are dummy indices which we sum over, we can relabel them. In particular, upon exchanging μ and ν in the first and second terms, we get exactly the third and fourth terms, i.e.

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}[\partial_\nu A_\mu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\nu A^\mu]. \tag{16}$$

These two terms can easily be integrated by parts, which gives (up to boundary terms)

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \xrightarrow{\text{IbP}} -\frac{1}{2}[-A_\mu \partial^2 A^\mu + A_\mu \partial_\nu \partial^\mu A^\nu] = \frac{1}{2}A_\mu [g^{\mu\nu}\partial^2 - \partial^\mu \partial^\nu] A_\nu. \tag{17}$$

where $\partial^2 \equiv \partial_\nu \partial^\nu$. Combining this expression with the rest of the Lagrangian gives the desired result.

2. *Find the propagator $D_{\nu\lambda}(k)$ from*

$$[-(k^2 - m^2) g^{\mu\nu} + k^\mu k^\nu] D_{\nu\lambda}(k) = \delta^\mu_\lambda \tag{18}$$

by using the fact that it has to be symmetric in the indices and making an ansatz made of the possible symmetric pieces.

Let us make the following Ansatz:

$$D_{\nu\lambda}(k) = A(k^2) g_{\nu\lambda} + B(k^2) k_\nu k_\lambda. \tag{19}$$

This is the most general expression one can start with upon imposing symmetry in the indices for a function that only depends on the vector k_μ. The functions A and B have all indices contracted, and thus depend only on k^2. Let us plug this in the equation of motion. We get

$$\begin{aligned} \delta^\mu_\lambda &= [-(k^2 - m^2) g^{\mu\nu} + k^\mu k^\nu][A(k^2) g_{\nu\lambda} + B(k^2) k_\nu k_\lambda] \\ &= -(k^2 - m^2) A \delta^\mu_\lambda + A k^\mu k_\lambda - (k^2 - m^2) B k^\mu k_\lambda + B k^2 k^\mu k_\lambda \\ &= -(k^2 - m^2) A \delta^\mu_\lambda + k^\mu k_\lambda [A + m^2 B]. \end{aligned} \tag{20}$$

The first term tells us that $A(k^2) = \frac{-1}{k^2 - m^2}$ while the second one must vanish. This implies $B = \frac{1}{m^2(k^2 - m^2)}$. We thus conclude that

$$D_{\nu\lambda}(k) = \frac{1}{k^2 - m^2} \left(-g_{\nu\lambda} + \frac{k_\nu k_\lambda}{m^2} \right). \tag{21}$$

Note that here again, an $i\epsilon$-prescription is usually understood in the denominator.

III. Continuous symmetry transformations

To put the formalism introduced in Sec. 2 to use, we apply it to the example of scale transformations*, where the coordinates are rescaled by a positive number λ.*

1. *The finite transformation acts as*

$$x \to \lambda x, \tag{22}$$
$$\Phi'(\lambda x) = \lambda^{-\Delta}\Phi(x), \tag{23}$$

where Δ is the so-called scaling dimension of the field. Find S'.

This a straightforward application of the formalism seen during the lectures. The Jacobian is given by

$$\left|\frac{\partial \vec{x}'}{\partial \vec{x}}\right| = \lambda^d, \tag{24}$$

which implies

$$S' = \lambda^d \int \mathrm{d}^d x\, \mathcal{L}\left(\lambda^{-\Delta}\Phi, \lambda^{-(\Delta+1)}\partial_\mu\Phi\right). \tag{25}$$

2. *The infinitesimal scale transformation acts as*

$$x \to x + \epsilon x \tag{26}$$
$$\mathcal{F}(\Phi) = (1 - \epsilon\Delta)\Phi. \tag{27}$$

Find the generator of the scale transformation.

We have $\frac{\delta x^\mu}{\delta\epsilon} = x^\mu$ and $\frac{\delta\mathcal{F}}{\delta\epsilon} = -\Delta$. By definition, generators are given by (see Eq. (2.12) of the notes)

$$iG_a\Phi = \frac{\delta x^\mu}{\delta\epsilon_a}\partial_\mu\Phi - \frac{\delta\mathcal{F}}{\delta\epsilon_a}. \tag{28}$$

Here, we shall call the generator of scale transformations D, which thus reads

$$\begin{aligned} D &= -i\left[\frac{\delta x^\mu}{\delta\epsilon}\partial_\mu - \frac{\delta\mathcal{F}}{\delta\epsilon}\right] \\ &= -ix^\mu\partial_\mu - i\Delta. \end{aligned} \tag{29}$$