Series 7

I. Canonical quantization of the free scalar field

1. *Verify that*

$$\varphi(t,\vec{x}) = \int \frac{\mathrm{d}^D k}{\sqrt{(2\pi)^D 2\omega_k}} [a(\vec{k}) e^{-i(\omega_k t - \vec{k}\cdot\vec{x})} + a^\dagger(\vec{k}) e^{i(\omega_k t - \vec{k}\cdot\vec{x})}] \tag{1}$$

with $\omega_k = \sqrt{\vec{k}^2 + m^2}$ *fulfills the field equation*

$$(\partial^2 + m^2)\varphi = 0. \tag{2}$$

This is rather straightforward. For convenience, we can rewrite Eq. (1) as

$$\varphi(t,\vec{x}) = \int \frac{\mathrm{d}^D k}{\sqrt{(2\pi)^D 2\omega_k}} \left[a(\vec{k}) e^{-ikx} + h.c.\right]_{k_0=\omega_k}, \tag{3}$$

where $h.c.$ stands for "Hermitean conjugate", and the expression inside the square brackets is evaluated on-shell, i.e. $k_0 = \omega_k \equiv \sqrt{\vec{k}^2 + m^2}$. When the Laplacian ∂^2 is applied to this expression, it only affects the exponentials, thus bringing some k^2 down. Since k is taken on-shell, it satisfies the equation of motion. Explicitly,

$$(\partial^2 + m^2)\varphi = \int \frac{\mathrm{d}^D k}{\sqrt{(2\pi)^D 2\omega_k}} \left[((-ik)^2 + m^2) a(\vec{k}) e^{-ikx} + h.c.\right]_{k_0=\omega_k} = 0 \tag{4}$$

because $-k^2 + m^2 = -\omega_k^2 + \vec{k}^2 + m^2 = 0$.

2. *Verify that*

$$[a(\vec{k}), a^\dagger(\vec{k}')] = \delta^{(D)}(\vec{k} - \vec{k}') \tag{5}$$

implies the canonical commutation relation

$$[\pi(t,\vec{x}), \varphi(t,\vec{x}')] = -i\delta^{(D)}(\vec{x} - \vec{x}'). \tag{6}$$

Starting from Eq. (3), we find that $\pi \equiv \dot{\varphi}$ is given by

$$\pi(t,\vec{x}) = \frac{-i}{2} \int \frac{\mathrm{d}^D k}{\sqrt{(2\pi)^D}} \sqrt{2\omega_k} \left[a(\vec{k}) e^{-ikx} - h.c.\right]_{k_0=\omega_k}. \tag{7}$$

Now, plugging Eq. (3) and (7) in an equal-time commutator, we expect four commutators of the form $[a,a]$, $[a^\dagger, a]$, $[a, a^\dagger]$ and $[a^\dagger, a^\dagger]$. But the first and the last vanish, so let us write explicitly the non-vanishing ones only:

$$= -\frac{i}{2} \iint \frac{\mathrm{d}^D k \mathrm{d}^D k'}{(2\pi)^D} \sqrt{\frac{2\omega_k}{2\omega_{k'}}} \Big[[a(\vec{k}), a^\dagger(\vec{k}')] e^{-ikx + ik'x'} - [a^\dagger(\vec{k}), a(\vec{k}')] e^{ikx - ik'x'}\Big]_{\text{on-shell}}, \tag{8}$$

where both k and k' are taken on-shell in the square brackets. Using Eq. (5), we can greatly simplify this expression as follows:

$$\begin{aligned} &= -\frac{i}{2}\int \frac{\mathrm{d}^D k}{(2\pi)^D}\left[e^{ik(x'-x)} + e^{ik(x-x')}\right]_{\text{on-shell}} \\ &= -i\int \frac{\mathrm{d}^D k}{(2\pi)^D} e^{i\vec{k}(\vec{x}-\vec{x}')} \\ &= -i\delta^{(D)}(\vec{x}-\vec{x}'), \end{aligned} \tag{9}$$

where, from the first line to the second one, we used the fact that $k(x'-x) = -\vec{k}(\vec{x}'-\vec{x})$ because x' and x are taken at the same time t, and we then changed $\vec{k} \to -\vec{k}$ in one of the exponentials. This is the canonical commutation relation of the field operators. However, Eq. (3) is not manifestly Lorentz invariant because of the measure. Instead, one often encounters the following measure:

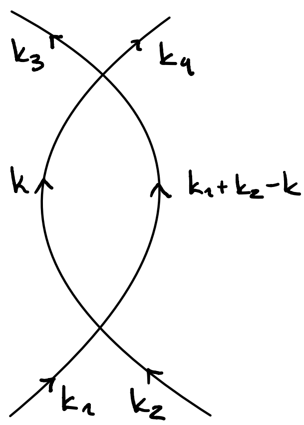
$$\mathrm{d}\Omega_k \equiv \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{1}{2\omega_k}, \tag{10}$$

where the factor of 2 is purely conventional. It can be shown that it is indeed invariant under (proper orthochronous) Lorentz transformations. In particular, we've already used it in the Feynman propagator and concluded that the latter was Lorentz invariant, as needed. Moreover, if one wants to normalize the field operators ϕ and π so as to integrate their Fourier modes over this measure, one needs to normalize the ladder operators in the appropriate way, namely:

$$[a(\vec{k}), a^\dagger(\vec{k}')] = 2\omega_k \cdot (2\pi)^D \delta^{(D)}(\vec{k}-\vec{k}'). \tag{11}$$

II. Loop amplitude from the canonical formalism

Calculate the amplitude of the process shown in the figure using the canonical formalism.



Let us first introduce the theoretical framework needed for this computation. The *scattering operator* S is defined as the limit of the time-evolution operator $U_I(t,t')$ in the interaction picture, i.e. $S = U_I(+\infty,-\infty)$ where $U_I(t,t')$ satisfies

$$i\frac{\partial}{\partial t}U_I(t,t') = H_I(t)U_I(t,t'), \tag{12}$$

with $H_I(t) = \int \mathrm{d}^D x\, \mathcal{H}_I$ the interaction part of the full Hamiltonian H. Solving this equation is non-trivial in the interaction picture, but if the interaction can be considered to be sufficiently small, one can solve it iteratively to get

$$U(t,t') = T\left[e^{-i\int_{t'}^{t}\mathrm{d}t''\, H_I(t'')}\right], \tag{13}$$

where T stands for the *time-ordered product* of the operators it acts on, which means they are arranged from left to right so that $t_1 > t_2 > \ldots > t_n$. Moreover, if we consider the (large, but not universal) class of theories for which the interaction does not contain derivatives of the fields, we can write the interaction Lagrangian density as $\mathcal{L}_I = -\mathcal{H}_I$, and we thus have

$$S = T\left[e^{i\int \mathrm{d}^{D+1}x\, \mathcal{L}_I}\right]. \tag{14}$$

This operator describes the time evolution of free particle states $|i\rangle$ coming from the far past into free particle states in the remote future $|f\rangle$ where some non-trivial scattering may have occurred at some finite time. Such states are created out of the free theory vacuum $|0\rangle$ upon acting with creation operators. For any such pair of states, we define an element of the *S-matrix* as $\langle f|\, S\, |i\rangle$.

To be more concrete, let $|i\rangle = a^\dagger(\vec{k}_i)\ldots a^\dagger(\vec{k}_1)\,|0\rangle \equiv |k_i,\ldots,k_1\rangle$ and $|f\rangle = a^\dagger(\vec{k}_{i+1})\ldots a^\dagger(\vec{k}_j)\,|0\rangle \equiv |k_{i+1},\ldots,k_j\rangle$. We then define the *scattering amplitude* $\mathcal{M}(i\to f)$ through

$$\langle f|\, S\, |i\rangle = \mathbb{1} + \frac{(2\pi)^{D+1}\delta^{(D+1)}\left(\sum_{l=1}^{i} k_l - \sum_{l=i+1}^{j} k_l\right)}{\prod_{l=1}^{j}\sqrt{(2\pi)^D 2\omega_{k_l}}}\cdot \mathcal{M}(i\to f), \tag{15}$$

where all k_l's are understood to be on-shell. That is, the scattering amplitude encodes the non-trivial part of the scattering up to an overall momentum conservation and some normalization factor. One can access this quantity to any desired order in the coupling constant(s) upon expanding our master formula, Eq. (14). This typically generates a (time-ordered) bunch of fields in between creation and annihilation operators. For a scalar field, we typically face expressions like

$$\langle 0|\, a(\vec{k}_j)\ldots a(\vec{k}_{i+1})T[\phi(x_1)\ldots\phi(x_n)]a^\dagger(\vec{k}_i)\ldots a^\dagger(\vec{k}_1)\,|0\rangle\,. \tag{16}$$

Evaluating such an expression is made rather simple thanks to *Wick's theorem*, which states that it equals

$$\sum_{full} a(\vec{k}_j)\ldots a(\vec{k}_{i+1})\cdot\phi(x_1)\ldots\phi(x_n)\cdot a^\dagger(\vec{k}_i)\ldots a^\dagger(\vec{k}_1), \tag{17}$$

where the sum runs over all fully contracted expressions. But what are these contractions? Well, the contraction from below is simply the vacuum expectation value of the two contracted terms. In particular,

$$\begin{aligned}\underline{\phi(x)a^\dagger(\vec{k})} &\equiv \langle 0|\,\phi(x)a^\dagger(\vec{k})\,|0\rangle = \left.\frac{e^{-ikx}}{\sqrt{(2\pi)^D 2\omega_k}}\right|_{k_0=\omega_k}\\ \underline{a(\vec{k})\phi(x)} &\equiv \langle 0|\,a(\vec{k})\phi(x)\,|0\rangle = \left.\frac{e^{ikx}}{\sqrt{(2\pi)^D 2\omega_k}}\right|_{k_0=\omega_k}.\end{aligned} \tag{18}$$

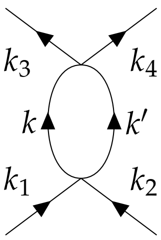
The one from above is the vacuum expectation value of the time-ordered product of the two contracted terms, e.g.

$$\phi(x)\phi(y) \equiv \langle 0| \, T[\phi(x)\phi(y)] \, |0\rangle = iD(x-y) = \int \frac{\mathrm{d}^{D+1}k}{(2\pi)^{D+1}} \frac{ie^{ik(x-y)}}{k^2 - m^2 + i\epsilon}. \quad (19)$$

Let us see how this machinery applies to the concrete case of a ϕ^4-interacting scalar theory, i.e. $\mathcal{L}_I = -\frac{\lambda}{4!}\phi^4$. To investigate the given diagram, we expand Eq. (14) to quadratic order in λ and we evaluate the S-matrix element $\langle k_4, k_3| \, S \, |k_2, k_1\rangle$:

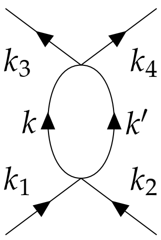
$$\langle k_4, k_3| \, S \, |k_2, k_1\rangle_{\lambda^2} = \frac{1}{2!}\left(\frac{-i\lambda}{4!}\right)^2 \iint_{xy} \langle 0| \, a(\vec{k}_4)a(\vec{k}_3) T[\phi^4(y)\phi^4(x)] a^\dagger(\vec{k}_2)a^\dagger(\vec{k}_1) \, |0\rangle \, , \quad (20)$$

where $\iint_{xy}$ stands for $\iint \mathrm{d}^{D+1}x\mathrm{d}^{D+1}y$. Among all the terms generated by Wick's theorem, only specific ones correspond to the diagram. They are the ones for which the incoming momenta k_1 and k_2 are connected to the first vertex at position x and the outgoing momenta k_3 and k_4 to the second vertex at position y (external legs), while the two vertices are connected together twice (internal legs). Typically,



$$\sim -\lambda^2 \iint_{xy} a(\vec{k}_4)a(\vec{k}_3)\phi(y)\phi(y)\phi(y)\phi(y)\phi(x)\phi(x)\phi(x)\phi(x)a^\dagger(\vec{k}_2)a^\dagger(\vec{k}_1).$$

But now, evaluating this expression is easy thanks to Eq. (18) and (19). Up to a symmetry factor, we have

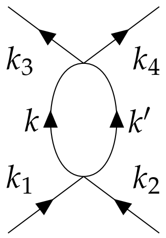


$$\sim -\lambda^2 \iint_{xy} \iint_{kk'} \frac{ie^{ik(x-y)}}{k^2 - m^2 + i\epsilon} \frac{ie^{ik'(x-y)}}{k'^2 - m^2 + i\epsilon} \left[\frac{e^{iy(k_3+k_4)}e^{-ix(k_1+k_2)}}{\prod_{i=1}^4 \sqrt{(2\pi)^D 2\omega_{k_i}}} \right]_{\text{on-shell}} , \quad (21)$$

where $\iint_{kk'}$ stands for $\iint \frac{\mathrm{d}^{D+1}k\mathrm{d}^{D+1}k'}{(2\pi)^{2(D+1)}}$ and all k_i's in the square bracket are on-shell. Note that the only x- and y-dependence is in the exponentials, so that

$$\iint_{xy} \frac{e^{ix(k+k'-k_1-k_2)}e^{iy(k_3+k_4-k-k')}}{(2\pi)^{2(D+1)}} = \delta^{(D+1)}(k+k'-k_1-k_2)\delta^{(D+1)}(k_1+k_2-k_3-k_4). \quad (22)$$

We further integrate over k' to get



$$\sim \frac{(2\pi)^{D+1}\delta^{(D+1)}(k_1+k_2-k_3-k_4)}{\prod_{i=1}^{4}\sqrt{(2\pi)^D 2\omega_{k_i}}}$$

$$\times(-\lambda)^2 \int \frac{\mathrm{d}^{D+1}k}{(2\pi)^{D+1}} \frac{i}{k^2-m^2+i\epsilon} \frac{i}{(k_1+k_2-k)^2-m^2+i\epsilon}, \quad (23)$$

where the k_i's are now understood implicitly to be on-shell. This leads us to the conclusion that the scattering amplitude corresponding to this diagram is given, up to a symmetry factor, by

$$\mathcal{M}(i\to f) \sim -\lambda^2 \int \frac{\mathrm{d}^{D+1}k}{(2\pi)^{D+1}} \frac{i}{k^2-m^2+i\epsilon} \frac{i}{(k_1+k_2-k)^2-m^2+i\epsilon} \quad (24)$$

which hopefully doesn't come as a surprise!

III. Charge of the free complex scalar field

Express the charge in terms of creation and annihilation operators:

$$Q = i\int \mathrm{d}^D x(\varphi^\dagger\partial_0\varphi - \partial_0\varphi^\dagger\varphi). \quad (25)$$

Let us first rewrite the charge as

$$Q = \int \mathrm{d}^D x \left[\phi^\dagger(t,\vec{x})\cdot i\pi(t,\vec{x}) + h.c.\right] \quad (26)$$

where we have

$$\begin{cases} \varphi^\dagger(t,\vec{x}) & = \int \frac{\mathrm{d}^D k}{\sqrt{(2\pi)^D 2\omega_k}} \left[a^\dagger(\vec{k})e^{ikx} + b(\vec{k})e^{-ikx}\right]_{k_0=\omega_k} \\ i\pi(t,\vec{x}) & = \frac{1}{2}\int \frac{\mathrm{d}^D k}{\sqrt{(2\pi)^D}}\sqrt{2\omega_{k'}}\left[a(\vec{k}')e^{-ik'x} - b^\dagger(\vec{k}')e^{ik'x}\right]_{k'_0=\omega_{k'}} . \end{cases} \quad (27)$$

The charge thus reads

$$Q = \frac{1}{2}\int \mathrm{d}^D x \iint \frac{\mathrm{d}^D k \mathrm{d}^D k'}{(2\pi)^D}\sqrt{\frac{2\omega_{k'}}{2\omega_k}}\cdot\left[I(t,\vec{x},\vec{k},\vec{k}') + h.c.\right], \quad (28)$$

where

$$\begin{aligned} I(t,\vec{x},\vec{k},\vec{k}') = {} & a^\dagger(\vec{k})a(\vec{k}')e^{i[t(\omega_k-\omega_{k'})-\vec{x}(\vec{k}-\vec{k}')]} - b(\vec{k})b^\dagger(\vec{k}')e^{-i[t(\omega_k-\omega_{k'})-\vec{x}(\vec{k}-\vec{k}')]} \\ & - a^\dagger(\vec{k})b^\dagger(\vec{k}')e^{i[t(\omega_k+\omega_{k'})-\vec{x}(\vec{k}+\vec{k}')]} + b(\vec{k})a(\vec{k}')e^{-i[t(\omega_k+\omega_{k'})-\vec{x}(\vec{k}+\vec{k}')]}. \end{aligned} \quad (29)$$

Integration over $\vec{x}$ turns part of these exponentials into δ-functions:

$$\int \frac{\mathrm{d}^D x}{(2\pi)^D} e^{i\vec{x}(\vec{k}\pm\vec{k}')} = \delta^{(D)}(\vec{k}\pm\vec{k}'), \quad (30)$$

which, in turn, kills an integration over the momentum. We are left with

$$Q = \frac{1}{2} \int \mathrm{d}^D k \left[\left\{ a^\dagger(\vec{k}) a(\vec{k}) - b(\vec{k}) b^\dagger(\vec{k}) \right\} + h.c. \right] \\ + \left[\left\{ b(\vec{k}) a(-\vec{k}) e^{-2i\omega_k t} - a^\dagger(\vec{k}) b^\dagger(-\vec{k}) e^{2i\omega_k t} \right\} + h.c \right] . \tag{31}$$

The first curly bracket is obviously a real expression and adding the hermitean conjugate merely doubles it. The second curly bracket, however, is purely imaginary and vanishes against the hermitean conjugate. Indeed,

$$\left[b(\vec{k}) a(-\vec{k}) e^{-2i\omega_k t} \right]^\dagger = a^\dagger(-\vec{k}) b^\dagger(\vec{k}) e^{2i\omega_k t} \tag{32}$$

and, upon replacing $\vec{k} \to -\vec{k}$, this kills the second term in the curly bracket, and vice-versa. We finally get

$$Q = \int \mathrm{d}^D k \left[a^\dagger(\vec{k}) a(\vec{k}) - b(\vec{k}) b^\dagger(\vec{k}) \right] . \tag{33}$$

We often see the normal-ordered version of this expression, which means that creation operators are placed on the left of annihilation operators. The price to pay to exchange b and $b^\dagger$ is an infinite shift coming from their commutation relation integrated over the whole space. As usual in QFT, we don't bother too much about such shifts and one should therefore not be surprised to see the charge written as

$$Q = \int \mathrm{d}^D k \left[a^\dagger(\vec{k}) a(\vec{k}) - b^\dagger(\vec{k}) b(\vec{k}) \right] . \tag{34}$$