The following notes are inspired by and follow closely Ch. 2 and 3 in Sidney Coleman's book.

Constructing a scalar quantum field

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8. QF as the main building block (Klein-Gordon, etc.)

Creation and annihilation operators in momentum space define Fock space:

$$[a_{\vec{p}}, a_{\vec{p}'}] = [a^\dagger_{\vec{p}}, a^\dagger_{\vec{p}'}] = 0$$

$$[a_{\vec{p}}, a^\dagger_{\vec{p}'}] = \delta^3(\vec{p}-\vec{p}')$$

unique and normalised: $\langle 0|0\rangle = 1$

Vacuum state $|0\rangle$: $a_{\vec{p}}|0\rangle = 0$

$$|\vec{p}\rangle = a^\dagger_{\vec{p}}|0\rangle, \quad |\vec{p}_1, \vec{p}_2\rangle = a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_2}|0\rangle, \text{ etc.}$$

Hamiltonian : $H = \int d^3p\, \omega_{\vec{p}}\, a^\dagger_{\vec{p}} a_{\vec{p}} = \int d^3p\, \omega_p \cdot N_{\vec{p}}$

$$\vec{P} = \int d^3p\, \vec{p}\, a^\dagger_{\vec{p}} a_{\vec{p}} = \int d^3p\, \vec{p}\, N_{\vec{p}}$$

where $N_{\vec{p}} = a^\dagger_{\vec{p}} a_p$ is the number (or density) operator, which counts the number of particles of a given momentum (integrating over $\vec{p}$ will give us the total number of particles).

Let's check that we understand how states and operators are defined and work.

$$\langle \vec{p}|\vec{p}'\rangle = \langle 0|a_{\vec{p}}\, a^\dagger_{\vec{p}'}|0\rangle = \langle 0|[a_{\vec{p}}, a^\dagger_{\vec{p}'}]|0\rangle + \langle 0|a^\dagger_{\vec{p}'} \cancel{a_{\vec{p}}|0\rangle}$$

$$= \delta^3(\vec{p}-\vec{p}')\, \cancel{\langle 0|0\rangle}$$

$$H|\vec{p}\rangle = H a^\dagger_{\vec{p}}|0\rangle = \int d^3q\, \omega_{\vec{q}}\, a^\dagger_{\vec{q}} a_{\vec{q}}\, a^\dagger_{\vec{p}}|0\rangle =$$

$$= \int d^3q\, \omega_{\vec{q}}\, a^\dagger_{\vec{q}} \left\{ [a_{\vec{q}}, a^\dagger_{\vec{p}}]|0\rangle + a^\dagger_{\vec{p}} \cancel{a_{\vec{q}}|0\rangle} \right\} =$$

$$= \int d^3q\, \omega_{\vec{q}}\, \delta^3(\vec{q}-\vec{p})\, a^\dagger_{\vec{q}}|0\rangle = \omega_{\vec{p}}|\vec{p}\rangle \quad \checkmark$$

$H|\vec{p}_1\vec{p}_2\rangle = ?$

$$H a^\dagger_{\vec p_1} a^\dagger_{\vec p_2} |0\rangle = \int d^3q \, \omega_{\vec q} \, a^\dagger_{\vec q} a_{\vec q} a^\dagger_{\vec p_1} a^\dagger_{\vec p_2} |0\rangle =$$

$$= \int d^3q \, \omega_{\vec q} \, a^\dagger_{\vec q} \left[\delta^3(\vec q - \vec p_1) + a^\dagger_{\vec p_1} a_{\vec q} \right] a^\dagger_{\vec p_2} |0\rangle =$$

$$= \int d^3q \, \omega_{\vec q} \, a^\dagger_{\vec q} \left[\delta^3(\vec q - \vec p_1) a^\dagger_{\vec p_2} + a^\dagger_{\vec p_1} \left[\delta^3(\vec q - \vec p_2) + \cancelto{0}{a^\dagger_{\vec p_2} a_{\vec q}} \right] \right] |0\rangle$$

$$= \left(\omega_{\vec p_1} + \omega_{\vec p_2} \right) a^\dagger_{\vec p_1} a^\dagger_{\vec p_2} |0\rangle \qquad \checkmark$$

Similarly for momentum operator.

Relativistic normalization and transformations:

In the definition of the operators so far we have ignored Lorentz invariance. For example in N or H.

The point is that d^3p is a rotationally invariant but not Lorentz invariant measure. d^4p is Lorentz invariant, but as we integrate over all momenta for a particle, we should always satisfy $p^2 = m^2$, and $p_0 > 0$. Solution:

$$\int d^4p \cdot \delta(p^2 - m^2) \, \theta(p_0) = \int d^3p \cdot \int dp_0 \, \delta\Big(p_0^2 - \underbrace{(\vec p^{\,2} + m^2)}_{\omega_{\vec p}^2}\Big) \theta(p_0)$$

$$= \int d^3p \int dp_0 \, \delta\big((p_0 - \omega_{\vec p})(p_0 + \omega_{\vec p})\big) \, \theta(p_0) =$$

$$= \int d^3p \int \frac{dp_0}{p_0 + \omega_{\vec p}} \, \delta(p_0 - \omega_{\vec p}) =$$

$$= \int \frac{d^3p}{2\omega_{\vec p}}$$

For convenience we multiply this by $\frac{1}{(2\pi)^3}$: $\displaystyle\int \frac{d^3p}{(2\pi)^3 2\omega_{\vec p}}$

Lorentz covariant states are therefore so defined:

$$|p\rangle = (2\pi)^{3/2}\sqrt{2\omega_{\vec{p}}}\,|\vec{p}\rangle$$

$$\Rightarrow \int d^3p\,\langle\vec{p}|\vec{p}'\rangle = \int d^3p\,\delta^3(\vec{p}-\vec{p}') = 1$$

$$\int\frac{d^3p}{(2\pi)^3 2\omega_{\vec{p}}}\langle p|p'\rangle = \int\frac{d^3p}{(2\pi)^3 2\omega_{\vec{p}}}(2\pi)^3\sqrt{2\omega_{\vec{p}}}\sqrt{2\omega_{\vec{p}'}}\,\delta^3(\vec{p}-\vec{p}') = 1$$

We can now define Lorentz and Poincaré transformations for these states.

First: translations and rotations:

$$U(\vec{a}) = e^{i\vec{p}\cdot\vec{a}} \qquad e^{i\vec{p}\cdot\vec{a}}|\vec{x}\rangle = |\vec{x}+\vec{a}\rangle;$$ Operator transformation:

$$O \to O' = U(\vec{a})^\dagger O\, U(a)$$

$U(R)$, where $R:\ \vec{p}\,' = R\vec{p}$ $\quad$ R a 3×3 orthogonal matrix

$$U(R)|\vec{p}\rangle = |R\vec{p}\rangle$$

$$U(R)U(R)^\dagger = 1$$

$$U(1) = 1$$

$$U(R_1)U(R_2) = U(R_1R_2)$$

For Poincaré and Lorentz transformations we can proceed analogously.

First of all, spacetime translations: $\vec{p}\cdot\vec{a} \to -(H\cdot T - \vec{p}\cdot\vec{a}) =$

$$= -p\cdot a \qquad a^\mu = (T,\vec{a})$$

$$O \to O' = e^{ip\cdot a}\,O\,e^{-ipa}$$

$$O = O(t),\ a = (T,\vec{0}) \Rightarrow O' = O(t+T)$$

$$e^{iHT}Oe^{-iHT}$$

$$\frac{\partial}{\partial T}\left(e^{iHT}Oe^{-iHT}\right) = iH\left(e^{iHT}Oe^{-iHT}\right) - \left(e^{iHT}Oe^{-iHT}\right)iH$$

$$= i[H,O'] \Rightarrow -i\frac{\partial}{\partial T}O' = [H,O']$$

which is nothing but the equation for the time evolution of operators in the Heisenberg picture.

Lorentz transformations:

$$V^\mu \to V'^\mu = \Lambda^\mu{}_\nu V^\nu = (\Lambda V)^\mu$$

$$p' = \Lambda p$$

$$\mathcal{U}(\Lambda)|p\rangle = |\Lambda p\rangle \quad ;$$

With this different normalization, we have to adapt our definition of creation and annihilation operators:

$$a^{(\dagger)} \to \alpha^{(\dagger)} \qquad \alpha^\dagger(p) = (2\pi)^{3/2}\sqrt{2\omega_{\vec{p}}}\, a^\dagger_{\vec{p}}$$

$$\alpha^\dagger(p) = |p\rangle = (2\pi)^{3/2}\sqrt{2\omega_{\vec{p}}}\,|\vec{p}\rangle$$

Lorentz transf. of these operators:

$$\mathcal{U}(\Lambda)|p\rangle = |\Lambda p\rangle$$

$$\mathcal{U}(\Lambda)\alpha^\dagger(p)|0\rangle = \mathcal{U}(\Lambda)\alpha^\dagger(p)\mathcal{U}^\dagger(\Lambda)\underbrace{\mathcal{U}(\Lambda)|0\rangle}_{|0\rangle}$$

$$= \underbrace{\mathcal{U}(\Lambda)\alpha^\dagger(p)\mathcal{U}^\dagger(\Lambda)}_{\alpha^\dagger(\Lambda p)}|0\rangle$$

$$\mathcal{U}(\Lambda)|p,p_1,\dots,p_n\rangle = \mathcal{U}(\Lambda)\alpha^\dagger(p)|p_1,\dots,p_n\rangle = \mathcal{U}(\Lambda)\alpha^\dagger(p)\mathcal{U}^\dagger(\Lambda)\mathcal{U}(\Lambda)|p_1,\dots,p_n\rangle$$

$$= \alpha^\dagger(\Lambda p)\mathcal{U}(\Lambda)|p_1,\dots,p_n\rangle$$

$$= \alpha^\dagger(\Lambda p)\alpha^\dagger(\Lambda p_1)|p_2,\dots,p_n\rangle$$

$$= \dots$$

$$= |\Lambda p,\Lambda p_1,\dots,\Lambda p_n\rangle$$

Translation operator acting on the $\alpha(p)$:

$$P^\mu |p_1, \dots, p_n\rangle = (p_1 + \dots + p_n)^\mu |p_1, \dots, p_n\rangle$$

$$\mathcal{U}(a) = e^{ipa} \qquad \mathcal{U}(a)|0\rangle = |0\rangle$$

$$\mathcal{U}(a)|p_1, \dots, p_n\rangle = e^{i\sum_i p_i \cdot a} |p_1, \dots, p_n\rangle$$

$$\mathcal{U}(a)\, \alpha^+(p_1) |p_2, \dots, p_n\rangle =$$

$$= \mathcal{U}(a)\, \alpha^+(p_1)\, \mathcal{U}^+(a)\, \mathcal{U}(a) |p_2, \dots, p_n\rangle$$

$$\prod_i \mathcal{U}(a)\, \alpha^+(p_i)\, \mathcal{U}^+(a) |0\rangle$$

$$\Rightarrow \mathcal{U}(a)\, \alpha^+(p_i)\, \mathcal{U}^+(a) = e^{ip\cdot a} \alpha^+(p)$$

by taking the h.c. we obtain

$$\mathcal{U}(a)\, \alpha(p)\, \mathcal{U}^+(a) = e^{-ipa} \alpha(p)$$

$$e^{iPa} \alpha^+(p) e^{-iPa} = e^{ipa} \alpha^+(p) \; ; \quad e^{iPa} \alpha(p) e^{-iPa} = e^{-ipa} \alpha(p)$$

Construction of a QFT.

Measurements have to respect causality: regions of spacetime which are causally disconnected cannot influence each other.

Measurements $\leftrightarrow$ observables $\xleftrightarrow{Q}$ operators.

In order to make a Lorentz-invariant quantum theory we want to build operators which automatically respect causality:

$$\boxed{R_1} \qquad\qquad \boxed{R_2}$$

R_1 and R_2 causally disconnected: $x_1 \in R_1$ and $x_2 \in R_2$

$$\Rightarrow (x_1 - x_2)^2 < 0$$

$\mathcal{O}_i$ observable to be measured in $R_i \Rightarrow [\mathcal{O}_1, \mathcal{O}_2] = 0$

We will now define as "building block" a quantum field, i.e. an operator or, better, a set of N operators $\phi^a(x)$, $a=1,\dots,N$ which satisfy the following properties:

1. $[\phi^a(x), \phi^b(y)] = 0$ if $(x-y)^2<0$

2. $\phi^a(x) = \phi^a(x)^\dagger$

3. $e^{-iPy}\phi^a(x)\, e^{iPy} = \phi^a(x-y)$

4. $U^\dagger(\Lambda)\phi^a(x)U(\Lambda) = \phi^a(\Lambda^{-1}x)$ $\Rightarrow$ we are assuming that the ϕ^a are <u>scalar</u> operators, i.e. that they themselves do not get rotated by Λ. We will study other cases later.

We will try an explicit construction of such quantum fields in terms of creation and annihilation operators and as a first attempt we will assume that the ϕ's depend linearly on $a^{(\dagger)}$.

$$\phi^a(x) = \int d^3p \left[F^a(x)a_{\vec p} + G^a(x)a^\dagger_{\vec p}\right]$$

Now one could spend time to explain why we have a - sign in condition 3. and why in condition 4. Λ^{-1}. First of all both are consistent with our previous discussion (we had e^{-iPy} on the lhs of 3. and $U(\Lambda)$ on the lhs of 4.) and moreover is the usual issue, that if I make an active transform. of the states and then evaluate the exp. value of an operator: $\quad f(x) = \langle\psi|\phi(x)|\psi\rangle$: $\quad |\psi'\rangle = e^{-iPa}|\psi\rangle$

$$\langle\psi'|\phi(x)|\psi'\rangle = \langle\psi|e^{iPa}\phi(x)e^{-iPa}|\psi\rangle = \langle\psi|\phi'(x)|\psi\rangle$$

the latter has to be equal to the value of the original function $f(x)$ after a shift backward:

Let us first look at the linearity in $a, a^\dagger$, by considering $\phi(0)$:

$$\phi(0) = \int\frac{d^3p}{(2\pi)^3 2\omega_p}\left[f_p\,\alpha(p) + g_p\,\alpha^\dagger(p)\right]$$

Lorentz transf.

$$U(\Lambda)\,\phi(0)\,U^\dagger(\Lambda) = \phi(0)$$

$$\Rightarrow \phi(0) = \int\frac{d^3p}{(2\pi)^3 2\omega_p}\left[f_p\,\alpha(p) + g_p\,\alpha^\dagger(p)\right]$$

$$= \int\frac{d^3p}{(2\pi)^3 2\omega_p}\left[f_p\,U(\Lambda)\alpha(p)U^\dagger(\Lambda) + g_p\,U(\Lambda)\alpha^\dagger(p)U^\dagger(\Lambda)\right]$$

$$= \int\frac{d^3p}{(2\pi)^3 2\omega_p}\left[f_p\,\alpha(\Lambda p) + g_p\,\alpha^\dagger(\Lambda p)\right]$$

$\Lambda p = p'$

$$= \int\frac{d^3p}{(2\pi)^3 2\omega_p}\left[f_{\Lambda^{-1}p}\,\alpha(p) + g_{\Lambda^{-1}p}\,\alpha^\dagger(p)\right]$$

$$\Rightarrow f_{\Lambda^{-1}p} = f_p \;;\; g_{\Lambda^{-1}p} = g_p \Rightarrow f, g, \text{ no dep. on } p$$

Condition 3. or how to transform $\phi(0) \to \phi(x)$

$$\phi(x) = e^{iPx}\phi(0)e^{-iPx}$$

use the transf. properties of $\overset{(+)}{\alpha}(p)$ to get:

$$\phi(x) = e^{iPx}\int\frac{d^3p}{(2\pi)^3 2\omega_p}\left[f\,\alpha(p) + g\,\alpha^+(p)\right]e^{-iPx} =$$

$$= \int\frac{d^3p}{(2\pi)^3 2\omega_p}\left[f\,e^{-ipx}\alpha(p) + g\,e^{ipx}\alpha^+(p)\right].$$

$\to$ Now cond. 1. and 2.

What we have seen so far is that the q.f. ϕ can be written as:

$$\phi(x) = f\,\phi^{(+)}(x) + g\,\phi^{(-)}(x)$$

$$\phi^{(+)}(x) = \int\frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}}\,a_{\vec p}\,e^{-ipx}$$

$$\phi^{(-)}(x) = \int\frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_p}}\,a^+_{\vec p}\,e^{ipx}$$

Cond. 2. Hermiticity: $\quad \phi^{(+)}(x)^+ = \phi^{(-)}(x)$

$$\phi^1(x) = \phi^{(+)} + \phi^{(-)} \quad ; \quad \phi^2(x) = i\left(\phi^{(+)}(x) - \phi^{(-)}(x)\right)$$

A generic linear combination of the two has the form

$$\phi(x) = e^{i\vartheta}\phi^{(+)}(x) + e^{-i\vartheta}\phi^{(-)}(x).$$

Which of these fields satisfies condition 1.,

i.e. that $[\phi^i(x), \phi^j(y)] = 0 \quad \forall\ (x-y)^2 < 0$?

Let us first consider $[\phi^1(x), \phi^2(y)] =$

$$= i\,\underbrace{\cancel{[\phi^{(+)}(x), \phi^{(+)}(y)]}}_{0} - i\,\underbrace{\cancel{[\phi^{(-)}(x), \phi^{(-)}(y)]}}_{0} - i\,[\phi^{(+)}(x), \phi^{(-)}(y)] + i\,[\phi^{(-)}(x), \phi^{(+)}(y)]$$

$$= -i\left\{ \left[\phi^{(+)}(x), \phi^{(-)}(y)\right] - \left[\phi^{(-)}(x), \phi^{(+)}(y)\right] \right\}$$

$$\left[\phi^{(+)}(x), \phi^{(-)}(y)\right] = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{p}}}}\, e^{-ipx} \int \frac{d^3p'}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{p}'}}}\, e^{ip'y} \underbrace{\left[a_{\vec{p}}, a^{+}_{\vec{p}'}\right]}_{\delta^3(\vec{p}-\vec{p}')} =$$

$$= \int \frac{d^3p}{(2\pi)^3 2\omega_{\vec{p}}}\, e^{-ip(x-y)} \equiv \Delta_+(x-y)$$

Note that $\Delta_+(x)$ is a Lorentz scalar: $\Delta_+(x) = \Delta_+(\Lambda x)$

For the second commutator we have:

$$\left[\phi^{(-)}(x), \phi^{(+)}(y)\right] = -\int \frac{d^3p}{(2\pi)^3 2\omega_{\vec{p}}}\, e^{ip(x-y)} = -\Delta_+(y-x)$$

Does $\Delta_+(x)$ vanish if $x^2<0$?

$$\frac{\partial}{\partial x_0}\Delta_+(x) = \int \frac{d^3p}{(2\pi)^3}\left(\frac{-i}{2}\right) \cdot e^{-ipx} = -\frac{i}{2}\int \frac{dp\, \vec{p}^2}{(2\pi)^2}\, e^{-i\omega_p x_0} \cdot \int_{-1}^{1} dz\, e^{-ip.r.z}$$

$$= \frac{-i}{(2\pi)^2 r}\int_{-\infty}^{\infty} dp\, |\vec{p}|\, e^{i(pr-\omega_p t)} \neq 0$$

Since $\left[\phi^1(x), \phi^2(y)\right] = i\left[\Delta_+(x-y) + \Delta_+(y-x)\right]$ and $\Delta_+(x) \neq 0$ for $x^2<0$, then we cannot have two different solutions to cond. 1.

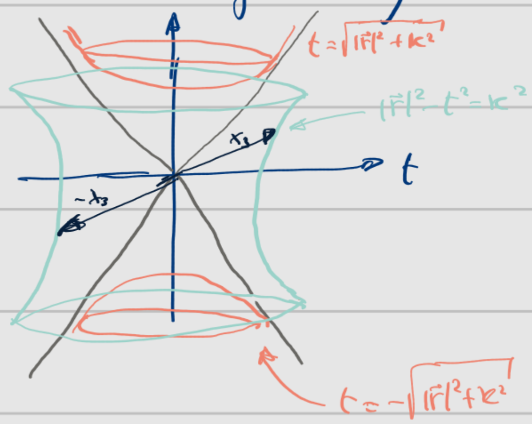
Now consider $\phi(x) = e^{i\theta}\phi^{(+)}(x) + e^{-i\theta}\phi^{(-)}(x)$ and first reabsorb $e^{-i\theta}$ in $a_{\vec{p}}$ and $e^{-i\theta}$ in $a^{+}_{\vec{p}}$

$$\Rightarrow \phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x),$$

$$\left[\phi(x), \phi(y)\right] = \left[\phi^{(+)}(x), \phi^{(-)}(y)\right] + \left[\phi^{(-)}(x), \phi^{+}(y)\right] = \Delta_+(x-y) - \Delta_+(y-x)$$

$$\equiv i\Delta(x-y)$$

But a spacelike vector can be transformed into its negative by a Lorentz transformation



$$\Rightarrow \quad \Delta_+(x-y)-\Delta_+(y-x)=\Delta_+(x-y)-\Delta_+(x-y)=0$$

$$\text{if } (x-y)^2<0$$

$$\Rightarrow \quad \phi(x)=\int\frac{d^3p}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{p}}}}\left[a_{\vec{p}}e^{-ipx}+a^+_{\vec{p}}e^{ipx}\right].$$

Let us now change point of view and start from the field ϕ, as a function (operator-valued) which satisfies the KG equation:

1. $$(\Box+m^2)\phi(x)=0$$

$$\phi(x)=\int\frac{d^4p}{(2\pi)^3}\,\delta(p^2-m^2)\,\theta(p^0)\left(\alpha(p)e^{-ipx}+\alpha^+(p)e^{ipx}\right)$$

$$(\Box+m^2)\phi(x)=\int\frac{d^4p}{(2\pi)^3}\underbrace{\delta(p^2-m^2)\theta(p^0)\left(-p^2+m^2\right)}_{\text{"0" everywhere}}\left(\alpha(p)e^{-ipx}+\alpha^+(p)e^{ipx}\right)$$

$$=0$$

2. $$[\phi(x),\phi(y)]=i\Delta(x-y)=\int\frac{d^3p}{(2\pi)^3 2\omega_p}\left[e^{-ip(x-y)}-e^{ip(x-y)}\right]=0 \quad \text{if } (x-y)^2<0$$

These two properties define $\phi(x)$ and from them one can derive the properties of the Fourier coefficients $a^{(+)}_{\vec{p}}$, in particular their commutation relations.

One final observation:

$[\phi(x), \phi(y)] = i\Delta(x-y)$ may be replaced by two equal-time commutation relations.

Reasoning:

$$(\Box_x + m^2)[\phi(x), \phi(y)] = (\Box_x + m^2)\, i\Delta(x-y) = 0$$

so, in order to determine $\Delta(x)$ from this commutation relation I need two initial conditions:

$$[\phi(\vec{x},t), \phi(\vec{y},t)] \text{ and } [\dot{\phi}(\vec{x},t), \phi(\vec{y},t)]$$

$$[\phi(\vec{x},t), \phi(\vec{y},t)] = \int \frac{d^3p}{(2\pi)^3 2\omega_p} \underbrace{\left(e^{i\vec{p}\cdot(\vec{x}-\vec{y})} - e^{-i\vec{p}(\vec{x}-\vec{y})}\right)}_{\text{odd function of } \vec{p}} = 0$$

odd function of $\vec{p}$; in polar coordinates $2i\sin(|\vec{p}||\vec{x}-\vec{y}|\cdot z)$ which integrated over $\int_{-1}^{1} dz$ gives zero.

$$[\dot{\phi}(\vec{x},t), \phi(\vec{y},t)] = \int \frac{d^3p}{(2\pi)^3 2\omega_p}\left[-i\omega_{\vec{p}}\, e^{i\vec{p}(\vec{x}-\vec{y})} - i\omega_{\vec{p}}\, e^{-i\vec{p}(\vec{x}-\vec{y})}\right]$$

$$= -i\int \frac{d^3p}{(2\pi)^3 2}\left[e^{i\vec{p}(\vec{x}-\vec{y})} + e^{-i\vec{p}(\vec{x}-\vec{y})}\right] = -i\int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}(\vec{x}-\vec{y})} = -i\delta^3(\vec{x}-\vec{y})$$