We concluded last week's lectures with the observation that, after having constructed the scalar field starting from creation and annihilation operators, we could forget about the latter and just rely on the following properties (or conditions) that a scalar quantum field had to satisfy:

1. $(\Box + m^2)\phi(x) = 0$
2. $[\phi(x), \phi(y)] = \int \frac{d^3p}{(2\pi)^3 2\omega_p}\left[e^{-ip(x-y)} - e^{ip(x-y)}\right] = 0 \qquad \text{if } (x-y)^2 < 0$

The latter can in fact be replaced by two equal-time commutation relations:

$$[\phi(\vec{x},t), \phi(\vec{y},t)] = 0$$

$$[\dot{\phi}(\vec{x},t), \phi(\vec{y},t)] = -i\,\delta^3(\vec{x}-\vec{y})$$

because the commutator also satisfies the KG equation

$$(\Box_x + m^2)[\phi(x), \phi(y)] = 0$$

and the equal-time commutators define the initial conditions which fix the solution of the differential equation completely.

Canonical quantization.

One can reach this conclusion taking a different path, namely that the step from classical to quantum is made by replacing Poisson brackets with commutators. This works for the mechanics of pointlike objects and we will assume that it works also for fields.

A system of particles (or a finite number of dofs) is described in the Hamiltonian formalism by N coordinate q^a and their conjugate momenta p^a, defined as $p_a = \frac{\partial L}{\partial \dot{q}^a}$, where

$$L = L(q^1,..,q^N, \dot{q}^1, ..., \dot{q}^N, t)$$

$$S = \int_{t_0}^{t_1} dt\, L(t) \qquad \boxed{\delta S = 0} \quad \text{for } q^a \to q^a + \delta q^a$$

EoM, which determine the trajectories $q^a(t)$

$$\delta S = \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial q^a} \cdot \delta q^a + \frac{\partial L}{\partial \dot{q}^a} \delta \dot{q}^a \right] = \int_{t_0}^{t_1} dt \underbrace{\left[\frac{\partial L}{\partial q^a} - \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{q}^a} \right]}_{\overset{\shortparallel}{0} \quad \text{Euler-Lagrange eqs.}} \cdot \delta q^a$$

In terms of p_a we get:

$$\delta S = \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial q^a} - \dot{p}_a \right] \cdot \delta q^a + \cancel{p_a \delta q^a \Big|_{t_0}^{t_1}}^{\,0 \text{ since } \delta q^a(t_0) = \delta q^a(t_1) = 0}$$

$$\Rightarrow \frac{\partial L}{\partial q^a} - \dot{p}_a = 0 \qquad \text{EoM}$$

Hamiltonian: $H \equiv p_a \cdot \dot{q}_a - L$ $\qquad H(q_a, p_a)$, so $\dot{q}_a$ has to be expressed in terms of the p_a (and q_a)

$$\delta H = \frac{\partial H}{\partial p_a} \cdot \delta p_a + \frac{\partial H}{\partial q^a} \cdot \delta q^a = \dot{q}^a \delta p_a + \underbrace{p_a \delta \dot{q}^a - \frac{\partial L}{\partial q^a} \delta q^a - \frac{\partial L}{\partial \dot{q}^a} \delta \dot{q}^a}_{}$$

$$= \dot{q}^a \delta p_a - \dot{p}_a \delta q^a$$

$$\Rightarrow \frac{\partial H}{\partial p_a} = \dot{q}^a \; ; \quad \frac{\partial H}{\partial q^a} = -\dot{p}_a$$

Poisson brackets: $\qquad \{F, G\} = \frac{\partial F}{\partial q^a} \frac{\partial G}{\partial p_a} - \frac{\partial F}{\partial p_a} \frac{\partial G}{\partial q^a}$

EoM

$$\frac{dF}{dt} = \{F, H\} + \frac{\partial F}{\partial t} = \frac{\partial F}{\partial q^a} \frac{\partial H}{\partial p_a} - \frac{\partial F}{\partial p_a} \frac{\partial H}{\partial q^a} + \frac{\partial F}{\partial t} = \frac{\partial F}{\partial q^a} \cdot \dot{q}_a + \frac{\partial F}{\partial p_a} \cdot \dot{p}_a + \frac{\partial F}{\partial t} \checkmark$$

$$\{q^i, q^j\} = \frac{\partial q^i}{\partial q^a}\cdot\cancel{\frac{\partial q^j}{\partial p_a}}^{\,0} - \cancel{\frac{\partial q^i}{\partial p_a}}^{\,0}\cdot\frac{\partial q^j}{\partial q^a} = 0$$

$$\{p_i, p_j\} = \cancel{\frac{\partial p_i}{\partial q^a}}^{\,0}\cdot\frac{\partial p_j}{\partial p_a} - \frac{\partial p_i}{\partial p_a}\cdot\cancel{\frac{\partial p_j}{\partial q^a}}^{\,0} = 0$$

$$\{q^i, p_j\} = \frac{\partial q^i}{\partial q^a}\,\frac{\partial p_j}{\partial p_a} - \cancel{\frac{\partial q^i}{\partial p_a}}^{\,0}\cancel{\frac{\partial p_j}{\partial q^a}}^{\,0} = \delta^i_a\cdot\delta^a_j = \delta^i_j$$

And from $\frac{dF}{dt} = \{F, H\}$

$$\dot{q}^i = \{q^i, H\} = \delta^i_a\,\frac{\partial H}{\partial p_a} = \frac{\partial H}{\partial p_i}$$

$$\dot{p}_i = \{p_i, H\} = -\delta^i_a\,\frac{\partial H}{\partial q^a} = -\frac{\partial H}{\partial q^i}$$

The principle of canonical quantization is that to make the step from a classical to a quantum theory one has to replace observables by operators and their Poisson brackets by commutators (multiplied by $-i$):

$$\boxed{\{\cdots,\cdots\} \longleftrightarrow -\frac{i}{\hbar}[\cdots,\cdots]}$$

(I will not show $\hbar$ in any of the following)

$$\{q^i, q^j\} = \{p_i, p_j\} = 0 \implies [q^i, q^j] = [p_i, p_j] = 0$$

$$\{q^i, p_j\} = \delta^i_j \implies [q^i, p_j] = i\delta^i_j \qquad \checkmark$$

The EoM for operators, for example, become:

$$\frac{dF}{dt} = -i[F, H] + \frac{\partial F}{\partial t}$$

This, when applied to q^i and p^i gives us:

$$\frac{dq^i}{dt} = -i[q^i, H] = -i\left(i\frac{\partial H}{\partial p_i}\right) = \frac{\partial H}{\partial p_i}$$

$$\frac{dp^i}{dt} = -i[p^i, H] = -i\left(-i\frac{\partial H}{\partial q_i}\right) = -\frac{\partial H}{\partial q_i}$$

$$[q^i, F] = \left[q^i, \sum_{\substack{n=0\\ m=0,n}}^{\infty} c_{a_1 \ldots a_m b_1 \ldots b_{m-n}}\, q^{a_1}\cdots q^{a_m}\, p_{a_{m+1}}\cdots p_{a_n}\right]$$

$$[q^i, q^{a_1}\cdots q^{a_m} p_{a_{m+1}}\cdots p_{a_n}] = q^{a_1}\cdots q^{a_m}[q^i, p_{a_{m+1}}\cdots p_{a_n}] =$$

$$= q^{a_1}\cdots q^{a_m}\left\{[q^i, p_{a_{m+1}}]p_{a_{m+2}}\cdots p_{a_n} + p_{a_{m+1}}[q^i, p_{a_{m+2}}]p_{a_{m+3}}\cdots p_{a_n} + \cdots + p_{a_{m+1}}\cdots[q^i, p_{a_n}]\right\} =$$

$$= q^{a_1}\cdots q^{a_n}\left\{i\delta^i_{a_{m+1}} p_{a_{m+2}}\cdots p_{a_n} + i\delta^i_{a_{m+2}} p_{a_{m+1}} p_{a_{m+3}}\cdots p_{a_n} + \cdots + p_{a_{m+1}}\cdots p_{a_{n-1}} i\delta^i_{a_n}\right\} =$$

$$q^{a_1}\cdots q^{a_m} i\frac{\partial}{\partial p_i}(p_{a_{m+1}}\cdots p_{a_n}) = i\frac{\partial}{\partial p_i}(q^{a_1}\cdots q^{a_m} p_{a_{m+1}}\cdots p_{a_n}) \checkmark$$

The step from classical mechanics to classical field theory entails the change from a finite number of discrete variables to an infinite number of continuous variables, but the formalism (after replacing sums with integrals etc.) and the EoMs look otherwise identical.

Classical field theory.

$$L = \int d^3x\, \mathcal{L}(\phi^a, \partial_\mu\phi^a, x)$$

$$S = \int_{t_1}^{t_2} dt\, L = \int d^4x\, \mathcal{L}(\phi^a, \partial_\mu\phi^a, x)$$

$$\delta S = 0 \Rightarrow \int d^4x\left[\frac{\partial\mathcal{L}}{\partial\phi^a}\cdot\delta\phi^a + \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}\cdot\delta(\partial_\mu\phi^a)\right]$$

$$= \int d^4x\left[\frac{\partial\mathcal{L}}{\partial\phi^a} - \partial_\mu\underbrace{\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}}_{\pi^\mu_a}\right]\delta\phi^a$$

$$\Rightarrow \int d^4x\left[\frac{\partial\mathcal{L}}{\partial\phi^a} - \partial_\mu\pi^\mu_a\right]\delta\phi^a = 0$$

$$\frac{\partial\mathcal{L}}{\partial\phi^a} = \partial_\mu\pi^\mu_a$$

Hamiltonian form:

$$H = \int d^3x \left(\pi_a \dot{\phi}^a - \mathcal{L} \right) \qquad ; \qquad \mathcal{H} = \pi_a \dot{\phi}^a - \mathcal{L}$$

Poisson brackets: $\displaystyle \{F, G\} = \int d^3x \left[\frac{\partial F}{\partial \phi^a(\vec{x})} \cdot \frac{\partial G}{\partial \pi_a(\vec{x})} - \frac{\partial F}{\partial \pi_a(\vec{x})} \frac{\partial G}{\partial \phi^a(\vec{x})} \right]$ all at time t

from which it follows

$$\{\phi^a(\vec{x},t), \phi^b(\vec{y},t)\} = 0 = \{\pi_a(\vec{x},t), \pi_b(\vec{y},t)\} \quad \text{since} \quad \frac{\partial \pi_a}{\partial \phi^b} = 0 = \frac{\partial \phi^a}{\partial \pi^b}$$

$$\{\phi^a(\vec{x},t), \pi_b(\vec{y},t)\} = \int d^3z \left[\delta^a_i \delta^3(\vec{x}-\vec{z}) \cdot \delta^i_b \delta^3(\vec{y}-\vec{z}) - 0 \right] = \delta^a_b \delta^3(\vec{x}-\vec{y})$$

$$\Downarrow$$

Canonical commutators:

$$[\phi^a(\vec{x},t), \phi^b(\vec{y},t)] = [\pi_a(\vec{x},t), \pi_b(\vec{y},t)] = 0$$

$$[\phi^a(\vec{x},t), \pi_b(\vec{y},t)] = i \delta_b \delta^3(\vec{x}-\vec{y})$$

What is $\pi_a(\vec{x},t)$? $\displaystyle \pi_a = \frac{\partial L}{\partial \dot{\phi}_a} = \dot{\phi}_a$

Take the free Lagrangian

$$\mathcal{L} = \frac{1}{2} \left(\partial_\mu \phi \, \partial^\mu \phi - m^2 \phi^2 \right) \longrightarrow \mathcal{H} = \frac{1}{2} \left(\pi^2 + |\vec{\nabla}\phi|^2 + m^2 \phi^2 \right)$$

$$\dot{\phi} = i[H, \phi(\vec{x},t)] = i \int d^3y \, \frac{1}{2} [\pi^2(\vec{y},t), \phi(\vec{x},t)] =$$

$$= i \int d^3y \, \frac{1}{2} \Big\{ \pi(\vec{y},t) \underbrace{[\pi(\vec{y},t), \phi(\vec{x},t)]}_{-i\delta^3(\vec{x}-\vec{y})} + \underbrace{[\pi(\vec{y},t), \phi(\vec{x},t)]} \pi(\vec{y},t) \Big\}$$

$$= \pi(\vec{x},t)$$

$$\dot{\pi} = i[H, \pi(\vec{x},t)] = i \int d^3y \, \frac{1}{2} \left\{ [|\vec{\nabla}\phi|^2, \pi] + m^2 [\phi^2, \pi] \right\}$$

$$= \vec{\nabla}^2 \phi(\vec{x},t) - m^2 \phi(\vec{x},t)$$

$$\Rightarrow \ddot{\phi}(\vec{x},t) = \vec{\nabla}^2\phi(\vec{x},t) - m^2\phi(\vec{x},t)$$

$$\Rightarrow (\Box + m^2)\phi = 0$$

This, together with the equal-time commutators completely define the system, and are identical to the conclusions we reached following the other path.

However, the present approach lends itself to generalizations to the case of a Lagrangian with interactions:

$$\mathcal{L} = \mathcal{L}_0 - \lambda\phi^4$$

Normal ordering.

Let's go back to the Hamiltonian:

$$\int d^3x \left[\pi(x)^2 + |\vec{\nabla}\phi(x)|^2 + m^2\phi^2(x) \right]$$

and express it in terms of the creation and annihilation operators, starting from:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_p}} \left[a_{\vec{p}}\, e^{-ipx} + a^\dagger_{\vec{p}}\, e^{ipx} \right]$$

$$\hookrightarrow \pi(x) = \dot{\phi}(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_{\vec{p}}}} (-i\omega_{\vec{p}}) \left[a_{\vec{p}}\, e^{-ipx} - a^\dagger_{\vec{p}}\, e^{ipx} \right]$$

After performing the triple integral one ends up with the following expression:

$$H = \frac{1}{2}\int d^3p\, \omega_{\vec p}\left[a^+_{\vec p} a_{\vec p} + a_{\vec p} a^+_{\vec p}\right]$$

which is not quite the same as $:H: \equiv \int d^3p\, \omega_{\vec p}\, a^+_{\vec p} a_{\vec p}$

we started with. The difference is:

$$H = \int d^3\vec p\, \omega_{\vec p}\left\{a^+_{\vec p} a_{\vec p} + \left[a_{\vec p}, a^+_{\vec p}\right]\right\} = :H: + \frac{1}{2}\int d^3p\, \omega_{\vec p}\, \delta^3(\vec 0)$$

$$= :H: + \underbrace{\delta^3(\vec 0)}_{\text{IR-div}} \cdot \underbrace{\frac{1}{2}\int d^3p\, \omega_{\vec p}}_{\text{UV-div}}$$

$$\delta^3(\vec p) = \int \frac{d^3x}{(2\pi)^3} e^{-i\vec p\vec x} \Rightarrow \delta^3(\vec 0) = \lim_{L\to\infty} \frac{1}{(2\pi)^3}\int_{-L/2}^{L/2} d^3x\; e^{i\vec p\cdot\vec x}\Big|_{\vec p=0}$$

$$\lim_{L\to\infty} \frac{L^3}{(2\pi)^3}$$

$$H = :H: + \frac{V}{(2\pi)^3}\cdot\frac{1}{2}\int d^3p\, \omega_{\vec p}$$

$$\mathcal{E}_0 \equiv \frac{E_0}{V} = \frac{1}{2(2\pi)^3}\int d^3p\, \omega_{\vec p}$$

is infinite because integrated up to $|\vec p| \to \infty$, where we don't really know whether we understand the physics.

$\Rightarrow$ We understand the origin of the infinities and that whether the vacuum energy is infinite or not is clearly beyond our control. The correctness of the theory, our construction of a QFT does not depend critically on this.

What we care about is the <u>difference</u> with respect to the vacuum energy, and this we can define by introducing the so-called

<u>normal ordering.</u>

$$:a_{\vec p_1}\cdots a_{\vec p_i}\,a^\dagger_{\vec q_1}\cdots a^\dagger_{\vec q_j}\,a_{\vec p_{i+1}}\cdots a_{\vec p_k}\,a^\dagger_{\vec q_{j+1}}\cdots: \;=\; a^\dagger_{\vec q_1}\cdots a^\dagger_{\vec q_m}\,a_{\vec p_1}\cdots a_{\vec p_n}$$

$$\Longrightarrow \quad :F:\,|0\rangle = 0$$

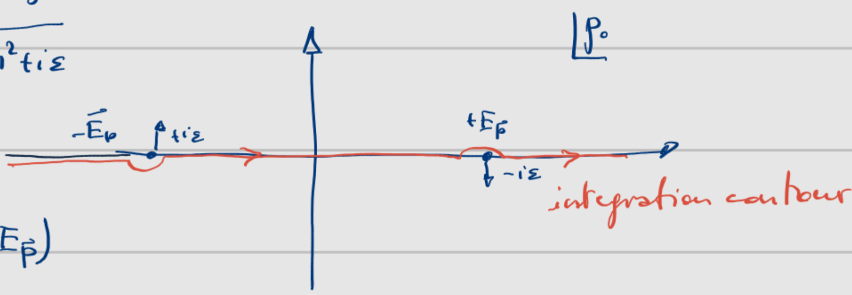
This explains the notation:

$$:H: = \int d^3p\, \omega_{\vec p}\, a^\dagger_{\vec p}\, a_{\vec p}$$

Feynman Propagator.

$$\Delta_F(x-y) = \langle 0|T\phi(x)\phi(y)|0\rangle \quad \text{where} \quad T\phi(x)\phi(y) = \begin{cases} \phi(x)\phi(y) & x_0 > y_0 \\ \phi(y)\phi(x) & x_0 < y_0 \end{cases}$$

$$\Delta_F(x-y) = i\int \frac{d^4p}{(2\pi)^4}\,\frac{e^{-ip(x-y)}}{p^2-m^2+i\varepsilon}$$



$$p^2-m^2 = (p^0-E_{\vec p})(p^0+E_{\vec p})$$

$$(p^2-m^2+i\varepsilon) = (p^0\underbrace{-E_{\vec p}+i\varepsilon}_{-(E_{\vec p}-i\varepsilon)})(p^0\underbrace{+E_{\vec p}-i\varepsilon}_{-(-E_{\vec p}+i\varepsilon)}) = p^{0\,2}-E_{\vec p}^2 + i\varepsilon(\cancel{p^0}+E_{\vec p}-\cancel{p^0}+E_{\vec p}) = p_0^2 - E_{\vec p}^2 + i\varepsilon$$

$$(\Box + m^2)\,\Delta_F(x-y) = 0$$