Complex scalar field.

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$$

$$\phi \equiv \frac{1}{\sqrt{2}} (\varphi_1 + i\varphi_2) \quad ;$$

EoM (Euler-Lagrange eqs. for the two fields present in the Lagrangian, ϕ and ϕ^*):

$$(\Box + m^2)\phi = 0$$

$$(\Box + m^2)\phi^* = 0$$

$$\phi = \int \frac{d^3p}{(2\pi)^3 \sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}} e^{-ipx} + b^+_{\vec{p}} e^{ipx} \right)$$

$$\phi^* = \int \frac{d^3p}{(2\pi)^3 \sqrt{2\omega_{\vec{p}}}} \left(b_{\vec{p}} e^{-ipx} + a^+_{\vec{p}} e^{ipx} \right)$$

Classical field momentum: $\quad \pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^*$

$$\pi = \int \frac{d^3p}{(2\pi)^3} \frac{(-i\omega_{\vec{p}})}{\sqrt{2\omega_{\vec{p}}}} \left(b_{\vec{p}} e^{-ipx} - a^+_{\vec{p}} e^{ipx} \right) = \int \frac{d^3p}{(2\pi)^3} (-i)\sqrt{\frac{\omega_{\vec{p}}}{2}} \left(b_{\vec{p}} e^{-ipx} - a^+_{\vec{p}} e^{ipx} \right)$$

$$\pi^+ = \int \frac{d^3p}{(2\pi)^3} \, i \sqrt{\frac{\omega_{\vec{p}}}{2}} \left(b^+_{\vec{p}} e^{ipx} - a_{\vec{p}} e^{-ipx} \right)$$

$$\left[\phi(\vec{x},t), \pi(\vec{y},t) \right] = i\,\delta^3(\vec{x}-\vec{y})$$

$$\left[\phi(\vec{x},t), \pi^+(\vec{y},t) \right] = 0$$

All others are either obtained by h.c. or zero.

These imply the following commutation relations for the creation and annihilation operators:

$$[a_{\vec{p}}, a^+_{\vec{p}'}] = (2\pi)^3 \delta^3(\vec{p}-\vec{p}') = [b_{\vec{p}}, b^+_{\vec{p}'}]$$

$$[a_{\vec{p}}, a_{\vec{p}'}] = [b_{\vec{p}}, b_{\vec{p}'}] = [a_{\vec{p}}, b_{\vec{p}'}] = [a_{\vec{p}}, b^\dagger_{\vec{p}'}] = 0$$

and h.c.

Classically conserved charge for a complex scalar field:

$$Q = i\int d^3x\,(\pi\phi - \phi^*\pi^*)$$

Quantum version thereof, with normal ordering:

$$:Q: = \int \frac{d^3p}{(2\pi)^3}\left(b^\dagger_{\vec{p}} b_{\vec{p}} - a^\dagger_{\vec{p}} a_{\vec{p}}\right) = N_b - N_a$$

$[H, Q] = 0 \implies Q$ is conserved. This will remain true even after adding interactions.

Nonrelativistic limit.

In the nonrelativistic limit $E_{\vec{p}} \simeq m + O(\vec{p}^2)$. To study how the formalism changes in this case it is useful to redefine the field after extracting the "main" time dependence:

$$\phi(x) = e^{-imt}\,\tilde{\phi}(\vec{x}, t)$$

KG equation for $\tilde{\phi}$:

$$(\Box + m^2)\phi = (\partial_t^2 - \Delta + m^2)\, e^{-imt}\tilde{\phi}(\vec{x},t) =$$

$$= e^{-imt}\left(\cancel{-m^2} - 2im\,\partial_t + \partial_t^2 - \Delta + \cancel{m^2}\right)\tilde{\phi}(\vec{x},t) = 0$$

$$\implies \left(\partial_t^2 - 2im\partial_t - \Delta\right)\tilde{\phi}(\vec{x},t) = 0$$

The time derivative of $\tilde{\phi}$ is proportional to the NR-energy, which is much smaller than m. So we can neglect ∂_t^2 wrt. $m\partial_t$:

$$\left(i\partial_t + \frac{\Delta}{2m}\right)\tilde{\phi}(\vec{x},t) = 0$$

which looks like the Schrödinger eq. for a free particle. However, $\tilde{\phi}$ is a field rather than a wavefunction. This field equation can be obtained from the following Lagrangian (to simplify the notation I am dropping the ~)

$$\mathcal{L} = i\phi^*\dot{\phi} - \frac{1}{2m}\vec{\nabla}\phi^*\cdot\vec{\nabla}\phi$$

Conserved current related to the invariance of the Lagrangian under the transformation $\phi \to e^{-i\alpha}\phi$:

$$j^\mu = \left(-\phi^*\phi, \frac{i}{2m}\phi^*\overleftrightarrow{\nabla}\phi\right)$$

Hamiltonian:

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = i\phi^*$$

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2m}\vec{\nabla}\phi^*\cdot\vec{\nabla}\phi$$

Canonical quantization:

$$\left[\phi(\vec{x}),\phi(\vec{y})\right] = \left[\phi^+(\vec{x}),\phi^+(\vec{y})\right] = 0$$

$$\left[\phi(\vec{x}),\phi^+(\vec{y})\right] = \delta^3(\vec{x}-\vec{y})$$

$$\phi(\vec{x}) = \int\frac{d^3p}{(2\pi)^3}\, a_{\vec{p}}\, e^{i\vec{p}\vec{x}}$$

with $\left[a_{\vec{p}}, a^+_{\vec{p}'}\right] = (2\pi)^3\,\delta^3(\vec{p}-\vec{p}')$

$|\vec{p}\rangle = a^+_{\vec{p}}|0\rangle$ etc. $\qquad \left[a_{\vec{p}},\phi^+(\vec{x})\right] = e^{-ipx}$

Energy of the one-particle state:

$$H|\vec{p}\rangle = \frac{\vec{p}^2}{2m}|\vec{p}\rangle$$

Notice that in order to solve the EoM for this system, or to satisfy the canonical quantization conditions, we only need one type of creation and annihilation operators, those for particles - Antiparticles have disappeared from the spectrum of the theory.

Recovering QM.

Momentum operator in QFT (or in Fock space):

$$\vec{P} = \int \frac{d^3p}{(2\pi)^3}\, \vec{p}\, a^\dagger_{\vec{p}} a_{\vec{p}}$$

Creation operator in position space:

$$\phi^\dagger(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\, a^\dagger_{\vec{p}}\, e^{-i\vec{p}\vec{x}} \Rightarrow |\vec{x}\rangle = \phi^\dagger(\vec{x})|0\rangle$$

Position operator (by analogy with the momentum operator):

$$\vec{X} = \int d^3x\, \vec{x}\, \phi^\dagger(\vec{x})\phi(\vec{x})$$

$$\vec{X}|\vec{x}\rangle = \vec{x}|\vec{x}\rangle$$

Generic state $|\psi\rangle$ obtained as superposition of $|\vec{x}\rangle$:

$$|\psi\rangle = \int d^3x\, \psi(\vec{x})|\vec{x}\rangle \; ;$$

$$\langle\psi|\psi\rangle = 1 \Rightarrow \int d^3x\, d^3y\, \psi^*(\vec{y})\psi(\vec{x}) \overbrace{\langle\vec{y}|\vec{x}\rangle}^{\delta^3(\vec{x}-\vec{y})} = \int d^3x\, |\psi(\vec{x})|^2 = 1 \,,$$

normal. condition for ψ

$$\vec{X}|\psi\rangle = \int d^3x\, \vec{x}\, \psi(\vec{x})|\vec{x}\rangle$$

$$\langle\psi|\vec{X}|\psi\rangle = \int d^3x\, d^3y\, \psi^*(y)\, \vec{x}\, \psi(\vec{x})\langle\vec{y}|\vec{x}\rangle = \int d^3x\, \vec{x}\, |\psi(\vec{x})|^2 \quad \checkmark$$

$$\vec{P}|\psi\rangle = \int \frac{d^3p}{(2\pi)^3}\, \vec{p}\, a^\dagger_{\vec{p}} a_{\vec{p}} \int d^3x\, \psi(\vec{x})|\vec{x}\rangle$$

$$\left[\; a_{\vec{p}}\,\phi^\dagger(\vec{x})|0\rangle = \int \frac{d^3q}{(2\pi)^3}\, \overbrace{a_{\vec{p}}\, a^\dagger_{\vec{q}}}^{(2\pi)^3\delta^3(\vec{p}-\vec{q})}\, e^{-i\vec{q}\vec{x}}|0\rangle = e^{-i\vec{p}\vec{x}}|0\rangle \;\right]$$

$$\Rightarrow \vec{P}|\psi\rangle = \int d^3x\, \psi(\vec{x}) \int \frac{d^3p}{(2\pi)^3}\, \vec{p}\, e^{-i\vec{p}\vec{x}}\, a^\dagger_{\vec{p}}|0\rangle =$$

$$= \int d^3x\, \psi(\vec{x})\, i\vec{\nabla}\phi^\dagger(\vec{x})|0\rangle = -i\int d^3x\, \vec{\nabla}\psi(\vec{x})\, \phi^\dagger(\vec{x})|0\rangle$$

$$= \int d^3x \left(\frac{1}{i}\vec{\nabla}\right)\psi(\vec{x})\,|\vec{x}\rangle$$

One can also easily show that:

$$[X^i, P^j]\,|\psi\rangle = i\delta^{ij}\,|\psi\rangle\;;$$

$$X^i P^j |\psi\rangle - X^i \int d^3x \left(\frac{1}{i}\partial^j\psi(\vec{x})\right)|\vec{x}\rangle - P^j \int d^3x\, \vec{x}\,\psi(\vec{x})|\vec{x}\rangle =$$

$$= \int d^3x \left(x^i \frac{1}{i}\partial^j - \frac{1}{i}\partial^j x^i\right)\psi(\vec{x})\,|\vec{x}\rangle = i\delta^{ij}\int d^3x\, \psi(\vec{x})|\vec{x}\rangle \quad \checkmark$$

Hamiltonian:

$$H = \int d^3x \frac{1}{2m}\vec{\nabla}\phi^* \cdot \vec{\nabla}\phi = \int \frac{d^3p}{(2\pi)^3}\frac{\vec{p}^{\,2}}{2m} a^\dagger_{\vec{p}} a_{\vec{p}} = \frac{\vec{P}^2}{2m}$$

$$i\frac{\partial}{\partial t}|\psi\rangle = \frac{\vec{P}^2}{2m}|\psi\rangle$$

$$\Longrightarrow \quad i\frac{\partial}{\partial t}\psi(\vec{x},t) = -\frac{\Delta}{2m}\psi(\vec{x},t),$$

i.e. the Schrödinger eq., and now for the wavefunction $\psi(\vec{x},t)$, which has a prob. interpret.

We can obtain the Schrödinger eq. with a potential if we include in the Lagrangian a term of the form $V(\vec{x})\phi^*\phi$:

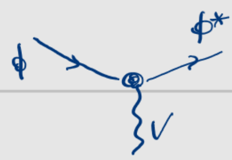
$$\mathcal{L} = i\phi^*\dot{\phi} - \frac{1}{2m}\vec{\nabla}\phi^* \cdot \vec{\nabla}\phi - V(\vec{x})\,\phi^*\phi$$

which, however, is not translation-invariant $\Rightarrow$ momentum is not conserved.

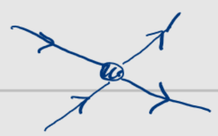
Interactions between particles can be obtained by adding terms like:

$$\Delta\mathcal{L} = \phi^*(\vec{x})\,\phi^*(\vec{x})\,\phi(\vec{x})\phi(\vec{x})$$

$V\phi^*\phi$



$\Delta\mathcal{L}$



Back to a relativistic setting: let us consider a generic Lagr. for an interacting field $\Rightarrow$ beyond a quadratic one.

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \sum_{n\geq 3}\frac{\tilde{\lambda}_n}{n!}\phi^n$$

Dimensional analysis:

$$[S] = [\hbar] = 1 \quad \Rightarrow \quad S = \int d^4x\, \mathcal{L} \quad \Rightarrow \quad [\mathcal{L}] = 4$$

$$\Rightarrow \quad [\phi] = 1$$

$$[\tilde{\lambda}_n] = 4 - n \quad \Rightarrow \quad \tilde{\lambda}_n = \frac{\lambda_n}{\Lambda^{n-4}} \quad \text{with } [\Lambda] = 1$$

$n = 3$: $\Lambda \cdot \lambda_3\, \phi^3$

dimensionless param: $\frac{\Lambda\lambda_3}{E}$

importance of this term depends on the energy scale Λ compared to the mass m and energy of the process considered

$\Longrightarrow$ relevant operator

$n = 4$: $\lambda_4\,\phi^4$

dimensionless param.: λ_4

marginal operator

$n > 4$ $\frac{\lambda_n\,\phi^n}{\Lambda^{n-4}}$

dimensionless param.: $\lambda_n\left(\frac{E}{\Lambda}\right)^{n-4}$

irrelevant operator

The classification of the operators according to their relevance refers to the limit $E \to 0$. In this course we will only consider interactions due to operators of the first and second type and handle them in perturbation theory, i.e. consider only the case $\lambda_n \ll 1$.

Examples:

1. $\lambda\phi^4$: $$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$

2. Scalar Yukawa theory:
$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - M^2\phi^*\phi - \frac{1}{2}m^2\varphi^2 - g\,\phi^*\phi\varphi$$
$$g \ll m, M$$

Interaction picture.

We have discussed QM in the Schrödinger picture, i.e. by letting the states carry the full time dependence:
$$i\frac{d}{dt}|\psi\rangle_S = H|\psi\rangle_S \quad\Rightarrow\quad |\psi(t)\rangle_S = e^{-iHt}|\psi(0)\rangle$$
whereas operators are constant in time: $\quad O_S : \frac{d}{dt}O_S = 0$
$$\langle\psi|O|\psi\rangle = {}_S\langle\psi|O_S|\psi\rangle_S = \langle\psi(0)|e^{iHt}O_S e^{-iHt}|\psi(0)\rangle$$
$$|\psi(0)\rangle \equiv |\psi\rangle_H = e^{iHt}|\psi\rangle_S$$
$$O_H \equiv e^{iHt}O_S e^{-iHt} \qquad \frac{d}{dt}O_H = i\left[H, O_H\right]$$
A third, in-between possibility is obtained if one splits the Hamiltonian in two pieces:
$$H = H_0 + H_{int}$$
$$|\psi(t)\rangle_I = e^{iH_0t}|\psi(t)\rangle_S$$
$$O_I(t) = e^{iH_0t}O_S e^{-iH_0t}$$

Since H_{int} is also an operator, and in general it does not commute with H_0, it will be time dependent. In the interaction picture it looks like:

$$H_I(t) = e^{iH_0 t} H_{int} e^{-iH_0 t}$$

Time evolution of states in the interaction picture:

$$i\frac{d}{dt}|\psi(t)\rangle_S = H_S|\psi(t)\rangle_S \Rightarrow i\frac{d}{dt}\left(e^{-iH_0 t}|\psi\rangle_I\right) = \left(H_0 + H_{int}\right)\left(e^{-iH_0 t}|\psi\rangle_I\right)$$

$$e^{-iH_0 t}\left(\cancel{H_0} + i\frac{d}{dt}\right)|\psi\rangle_I = e^{-iH_0 t}\left(\cancel{H_0} + H_I\right)|\psi\rangle_I$$

$$\Rightarrow \boxed{i\frac{d}{dt}|\psi\rangle_I = H_I|\psi\rangle_I}$$

states evolve according to the interaction Hamiltonian;

$$\boxed{\frac{d}{dt}O_I(t) = i\left[H_0, O_I\right]}$$

operators evolve according to the "free" Hamiltonian;

Dyson's formula

Solution of

$$i\frac{d}{dt}|\psi\rangle_I = H_I|\psi\rangle_I$$

$$|\psi(t)\rangle_I = U(t,t_0)|\psi(t_0)\rangle_I$$

$U(t,t_0)$ = unitary time-evolution operator ; $U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3)$

$U(t,t) = 1$

$$i\frac{d}{dt}U(t,t_0) = H_I(t)\,U(t,t_0)$$

$$U(t,t_0) = T\exp\left(-i\int_{t_0}^{t} dt'\, H_I(t')\right)$$

where T stands for time ordering.

The necessity of T can be seen as follows: if you just have the exponential, this is defined by its Taylor expansion, since its argument is an operator:

$$\exp\left(-i\int_{t_0}^{t} dt'\, H_I(t')\right) = 1 - i\int_{t_0}^{t} dt'\, H_I(t') - \frac{1}{2}\int_{t_0}^{t} dt' dt''\, H_I(t') H_I(t'') + \dots$$

Starting from the second term, the ordering of the operators is ambiguous, since they may not commute for $t' \neq t''$. That this is a real issue can be seen as follows:

If we take the time derivative of the latter we get:

$$-\frac{1}{2}\left(H_I(t)\cdot\int_{t_0}^{t} dt'\, H_I(t') + \int_{t_0}^{t} dt'\, H_I(t')\cdot H_I(t)\right)$$

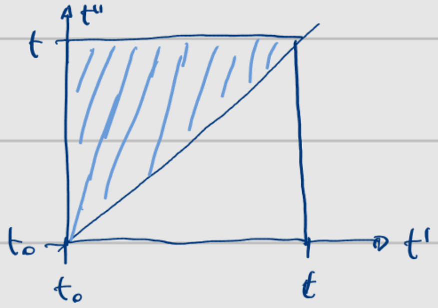
which is not quite what we get from the equation, which tells us that $H_I(t)$ should always be to the left of $U(t,t_0)$, (considering that the latter is obtained by resumming the series into an exponential again).

To exclude the second ordering we note that $t' \leq t$, so that if we want $H_I(t)$ to the left of the integral we obtain it by saying that operators at later times must be to the left of operators at earlier ones.

$$T\big(O_1(t_1)O_2(t_2)\big) = \begin{cases} O(t_1)\,O(t_2) & \text{if } t_1 > t_2 \\ O(t_2)\,O(t_1) & \text{if } t_2 > t_1 \end{cases}$$

Let us make very explicit how this works on the exponential:

$$U(t,t_0) = 1 - i\int_{t_0}^{t} dt'\, H_I(t') - \frac{1}{2}\int_{t_0}^{t} dt' \left\{\int_{t_0}^{t'} dt''\, H_I(t') H_I(t'') + \int_{t'}^{t} dt''\, H_I(t'') H_I(t')\right\} + \dots$$



$$\int_{t_0}^{t} dt' \int_{t'}^{t} dt'' H_I(t'') H_I(t') = \int_{t_0}^{t} dt'' \int_{t_0}^{t''} dt' H_I(t'') H_I(t')$$

$$= \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' H_I(t') H_I(t'')$$

which is equal to the first term.

$$U(t,t_0) = 1 - i\int_{t_0}^{t} dt' H_I(t') - \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' H_I(t') H_I(t'') + \dots$$

To show that this is indeed a solution of the equation we just need to observe that under the T sign all operators commute:

$$i\frac{\partial}{\partial t}\left\{T\exp\left(-i\int_{t_0}^{t} dt' H_I(t')\right)\right\} = T\left[H_I(t)\cdot\exp\left(-i\int_{t_0}^{t} dt' H_I(t')\right)\right]$$

$$= H_I(t)\, T\left[\exp\left(-i\int_{t_0}^{t} dt' H_I(t')\right)\right]$$

S-matrix.

In QFT we are mainly interested in calculating scattering and decay processes. We can describe these via initial states which we consider in the far past, in which all particles involved are far away from each other, isolated. Their time evolution brings them closer together, they interact briefly and move away from each other, flying as isolated particles. To describe these processes it is useful to introduce the

S-matrix, which is defined as follows.

Consider the evolution of a state $|\psi\rangle$ towards the far past. $\lim_{t\to-\infty} |\psi(t)\rangle$

This evolves according to the full Hamiltonian H, however, if in the past all particles involved in the state are far away from each other, they behave as if $H = H_0$. Formally we can formulate this in the following way:

given a state $|\psi(t)\rangle \in \mathcal{H}_0$ (Hilbert space for H_0), there is a state $|\psi(t)\rangle_{in} \in \mathcal{H}$ (Hilbert space for H) such that

$$\lim_{t\to-\infty} \left\| |\psi(t)\rangle - |\psi(t)\rangle_{in} \right\| = 0$$

and similarly for the $t\to\infty$ limit, for which we will call the relevant states of $\mathcal{H}$ $|\phi\rangle_{out}$:

$$\lim_{t\to\infty} \left\| |\phi(t)\rangle - |\phi(t)\rangle_{out} \right\| = 0$$

What we are interested to calculate is the probability that an in-state will evolve into an out-state into the future. The probability amplitude for such a process is given by:

$$_{out}\langle \phi | \psi \rangle_{in} \equiv \langle \phi | S | \psi \rangle$$

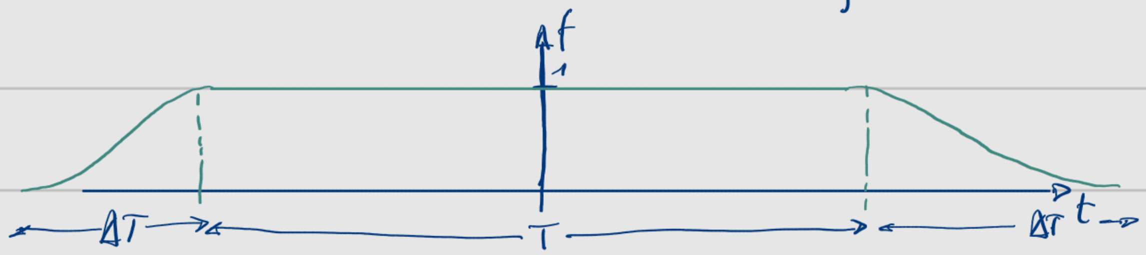
with S an operator called the S-matrix, which has the property of being unitary:

$$S^\dagger S = S S^\dagger = \mathbb{1}$$

and that it commutes with H_0: $[S, H_0] = 0$

Having introduced the time-evolution operator $U(t,t')$ we can make the connection between this and the S-matrix. To make this connection we need to introduce an approximation:

$$H = H_0 + H_I(t) \to H_0 + f(t,T,\Delta)\, H_I(t)$$



With this trick it is easy to obtain the following expression for the S-matrix:

$$S = U_I(\infty,-\infty)$$

which will be derived in the next lecture.