Let us derive the formula we have written down at the end of last week:

$$S = U_I(\infty, -\infty) \ ,$$

where $\;{}_{out}\langle\phi|\psi\rangle_{in} = \langle\phi|S|\psi\rangle\;$ with $|\phi\rangle_{out}, |\psi\rangle_{in} \in \mathcal{H}$ and $|\phi\rangle, |\psi\rangle \in \mathcal{H}_0$

and $$\lim_{t\to\pm\infty} \| e^{-iH_0 t}|\psi\rangle - e^{-iHt}|\psi\rangle_{\substack{out\\in}} \| = 0 \ .$$

The key observations now are the following:

- any $|\psi\rangle \in \mathcal{H}_0$ does not evolve in time in the interaction picture
- ${}_{out}\langle\phi|\psi\rangle_{in} = {}_{out}\langle\phi(t)|\psi(t)\rangle_{in}$ is independent of time since both states evolve according to the same Schrödinger equation,
- namely $\;|\psi(t)\rangle_{\substack{in\\out}} = U_I(t,t_0)\,|\psi(t_0)\rangle_{\substack{in\\out}}$
- having introduced the $f(t,T,\Delta T)$ weight function, H coincides with H_0 both in the far past as well as future. So, when we take the $t\to\pm\infty$ limit $\;|\psi(\pm\infty)\rangle_{\substack{out\\in}} \begin{smallmatrix}\nearrow |\phi\rangle \\ \searrow |\psi\rangle\end{smallmatrix} \in \mathcal{H}_0$: we do not need to worry about the different Hilbert spaces.

- $$\lim_{t_2\to\infty} {}_{out}\langle\phi(t_2)|\psi(t_2)\rangle_{in} = \lim_{t_2\to\infty} \langle\phi|U_I(t_2,t_1)|\psi(t_1)\rangle = \lim_{\substack{t_1\to-\infty\\t_2\to\infty}} \langle\phi|U_I(t_2,t_1)|\psi\rangle$$
$$= \langle\phi|U_I(\infty,-\infty)|\psi\rangle$$

The discussion of this point by Coleman I don't find particularly clear (rather confusing, in fact). The way I understand it is discussed below.

We first remind ourselves that we are working in the interaction picture, and that in this picture the time evolution of a state is given by:

$$|\psi(t)\rangle = e^{iH_0 t} e^{-iH(t-t_0)} e^{-iH_0 t_0} |\psi(t_0)\rangle,$$

which is obtained from:

$$|\psi(t)\rangle_s = e^{-iH(t-t_0)} |\psi(t_0)\rangle_s$$

$$|\psi(t_0)\rangle_s = e^{-iH_0 t_0} |\psi(t_0)\rangle$$

$$|\psi(t)\rangle = e^{iH_0 t} e^{-iH(t-t_0)} e^{-iH_0 t_0} |\psi(t_0)\rangle = U_I(t,t_0)|\psi(t_0)\rangle$$

$$|\psi(-\infty)\rangle_{in} = \lim_{t\to-\infty} U_I(t,0)|\psi\rangle_{in}$$

$|\psi(-\infty)\rangle_{in} = |\psi\rangle$ which is time independent, as a vector in the which, in the interaction picture does not evolve in time.

$$|\psi\rangle_{in} = \lim_{t\to-\infty} U_I(0,t)|\psi\rangle$$

and analogously

$$|\phi\rangle_{out} = \lim_{t\to\infty} U_I(0,t)|\phi\rangle$$

$$_{out}\langle\phi|\psi\rangle_{in} = \lim_{\substack{t'\to-\infty \\ t''\to\infty}} \langle\phi|U_I^\dagger(0,t'')U_I(0,t')|\psi\rangle = \lim_{\substack{t'\to-\infty \\ t''\to\infty}} \langle\phi|U_I(t'',0)U_I(0,t')|\psi\rangle$$

$$= \lim_{\substack{t'\to-\infty \\ t''\to\infty}} \langle\phi|U_I(t'',t')|\psi\rangle = \langle\phi|U_I(\infty,-\infty)|\psi\rangle$$

Wick's theorem.

Our goal is now to calculate matrix elements of the S-matrix:

$$S = U_I(\infty,-\infty) = T\exp\left[\int_{-\infty}^{\infty} dt'\, H_I(t')\right]$$

In a scalar field theory H_I will be given by one (or more) monomial. Like $g\,\phi^*\phi\varphi$ or $\lambda \frac{\phi^4}{4!}$, where H_I is taken as normal product (which I haven't explicitly written). For a scattering process like $2 \to 2$, we will have to evaluate a matrix element of the form:

$$\langle p_1' p_2' | :\phi^4(x_1) \dots \phi^4(x_n): | p_1 p_2 \rangle$$

for the second kind of theory. For the first it could be:

$$\langle \phi(p_1')\phi(p_2') | :\phi^*(x_1)\phi(x_1)\varphi(x_1) \dots \phi^*(x_n)\phi(x_n)\varphi(x_n): | \phi(p_1)\phi(p_2) \rangle$$

To simplify the calculation we can rely on Wick's theorem:

- we define the contraction of two fields as follows:

$$\overbrace{A(x)B(y)} = T(A(x)B(y)) - :A(x)B(y):$$

  and first prove that it is a c-number (not an operator).

- Consider $x_0 > y_0$, then

$$T(A(x)B(y)) = A(x)B(y)$$

$$A(x) = \underbrace{A^{(+)}(x)}_{a^-} + \underbrace{A^{(-)}(x)}_{a^+} \quad \text{and similarly for } B.$$

$$T(AB) = A^{(+)}(x)B^{(+)}(y) + \underbrace{A^{(+)}(x)B^{(-)}(y)}_{\text{not n.o.}} + A^{(-)}(x)B^{(+)}(y) + A^{(-)}(x)B^{(-)}(y)$$

$$\Rightarrow T(AB) = :A(x)B(y): + \left[A^{(+)}(x), B^{(-)}(y)\right]$$

but we know already that the commutator is a c-number:

$$\left[A^{(+)}(x), B^{(-)}(y)\right] = \begin{cases} \Delta_+(x-y) & \text{if } A=B \\ 0 & \text{if } A \neq B \end{cases}$$

We can reason similarly for the other case, $x_0 < y_0$.

So that, putting everything back together:

$$\overbracket{A(x)\,A(y)} = \theta(x_0-y_0)\left[A^{(+)}(x), A^{(-)}(y)\right] + \theta(y_0-x_0)\left[A^{(+)}(y), A^{(-)}(x)\right]$$

$$= \theta(x_0-y_0)\,\Delta_+(x-y) + \theta_F(y_0-x_0)\,\Delta_+(y-x)$$

If we take the v.e.v. of the contraction we get the contraction itself, since it is a c-number:

$$\overbracket{A(x)B(y)} = \langle 0|\,\overbracket{A(x)B(y)}\,|0\rangle = \langle 0|T(A(x)B(y))|0\rangle - \cancelto{0}{\langle 0|:A(x)B(y):|0\rangle}$$

$$= \langle 0|T\,A(x)\,B(y)\,|0\rangle$$

$\Rightarrow \quad \overbracket{\varphi(x)\varphi(y)} = \langle 0|T\varphi(x)\varphi(y)|0\rangle$, which is nothing but Feynman's propag.

$$\langle 0|T\varphi(x)\varphi(y)|0\rangle = \lim_{\varepsilon\to 0^+}\int\frac{d^4p}{(2\pi)^4}\,e^{-ip(x-y)}\,\frac{i}{p^2-m^2+i\varepsilon} \equiv \Delta_F(x-y)$$

For a complex field we get:

$$\overbracket{\phi^*(x)\phi(y)} = \overbracket{\phi(y)\phi^*(x)} = \Delta_F(x-y)$$

and all other contractions equal to zero (including those of $\phi^{(*)}$ and φ).

- Consider now the contraction of two field which appear in a monomial involving other fields also:

$$:A(x)\,\overbracket{B(y)\,C(z)\,D(w)}: \;\equiv\; :A(x)C(z):\;\overbracket{B(y)\,D(w)}$$

and introduce the notation $\phi_i = \phi(x_i)$.

- Wick's theorem:

$$T(\phi_1 \cdots \phi_n) = :\phi_1\cdots\phi_n: + :\text{all possible contractions}: \;\equiv W(\phi_1,\dots,\phi_n)$$

where, to be explicit:

$:$ all possible contractions $: \; = \; :\phi_1\phi_2\phi_3\cdots\phi_n: \; +$ (all other terms w/ 1 contr.)

$+ :\phi_1\phi_2\phi_3\phi_4\cdots\phi_n:$ (all other terms w/ 2 contr.)

$+ \dots$ (all terms w/ $\frac{n}{2}$ or $\frac{n-1}{2}$ contr.)

- Proof (by induction):

it is true by definition for $n=2$.

If it is true for $\phi_2\cdots\phi_n$ and we multiply it by ϕ_1 and consider first the case $x_1^0 > x_k^0 \quad \forall k=(2,\dots,n)$, we get:

$$T(\phi_1\phi_2\cdots\phi_n) = \left(\phi_1^{(+)} + \phi_1^{(-)}\right) W(\phi_2,\dots,\phi_n)$$

$\phi_1^{(+)}$ must move to the left of all $\phi_k^{(+)} \quad \forall k=2,\dots,n$, and thereby we get a commutator:

$$T(\phi_1\cdots\phi_n) = \underbrace{\phi_1^{(-)} W(\phi_2\cdots\phi_n)}_{\text{n.o.}} + \underbrace{W(\phi_2,\dots,\phi_n)\phi_1^{(+)}}_{\text{n.o.}} + \left[\phi_1^{(+)}, W(\phi_2,\dots,\phi_n)\right]$$

$\left[\phi_1^{(+)}, W(\phi_2,\dots,\phi_n)\right]$ is not yet n.o.

each term in $W(\phi_2,\dots,\phi_n)$ contains a number of contractions (= prop.) times n.o. fields. The propagators commute, so we can ignore them

$$\left[\phi_1^{(+)}, :\phi_2\cdots\phi_\ell:\right] = \left[\phi_1^{(+)}, \phi_2^{(-)}\cdots\phi_k^{(-)}\phi_{k+1}^{(+)}\cdots\phi_\ell^{(+)}\right] + \dots$$

$$= \left[\phi_1^{(+)}, \phi_2^{(-)}\right]\phi_3^{(-)}\cdots\phi_k^{(-)}\phi_{k+1}^{(+)}\cdots\phi_\ell^{(+)} +$$

$$+ \phi_2^{(-)}\left[\phi_1^{(+)}, \phi_3^{(-)}\right]\cdots\phi_k^{(-)}\phi_{k+1}^{(+)}\cdots\phi_\ell^{(+)} + \dots$$

$$= \Delta_{+,12} :\phi_3^{(-)}\cdots\phi_\ell^{(+)}: + \Delta_{+,13} :\phi_2^{(-)}\cdots\phi_\ell^{(+)}: + \dots$$

To move past each of the $\phi_k^{(-)}$ fields in the n.o. product $\phi_1^{(+)}$ generate one Δ_+ propagator, and once we consider the case $x_0 < y_0$

we have the opposite situation, which makes us generate all possible Δ_{F1k1}. Combining the two cases we generate all possible $\Delta_{F,1,k}$, i.e. all possible contractions, so that, all in all, we get first all terms where ϕ_1 is not involved in contractions, and then all possible terms where ϕ_1 is in all possible contractions.

If $T(\phi_2 \cdots \phi_n) = W(\phi_2, \dots, \phi_n) \Rightarrow T(\phi_1 \cdots \phi_n) = W(\phi_1, \dots \phi_n)$. QED.

<u>Example:</u> nucleon-nucleon scattering (in a $g\phi^*\phi\varphi$ theory)

$\phi\phi \to \phi\phi$

$$|i\rangle = \sqrt{2\omega_{\vec{p}_1}}\sqrt{2\omega_{\vec{p}_2}}\, b^\dagger_{\vec{p}_1} b^\dagger_{\vec{p}_2} |0\rangle \equiv |p_1 p_2\rangle$$

$$|f\rangle = \sqrt{2\omega_{\vec{p}_1'}}\sqrt{2\omega_{\vec{p}_2'}}\, b^\dagger_{\vec{p}_1'} b^\dagger_{\vec{p}_2'} |0\rangle \equiv |p_1' p_2'\rangle$$

$$\langle f|S-1|i\rangle = \langle f| \underbrace{T\exp\left[-i\int d^4x\, H_I(x)\right] - 1}_{-i\int d^4x\, g\, \phi^*\phi\varphi + \frac{(-ig)^2}{2}\int d^4x_1 d^4x_2\, T\left(\phi^*(x_1)\phi(x_1)\varphi(x_1)\phi^*(x_2)\phi(x_2)\varphi(x_2)\right) + \dots} |i\rangle$$

$$T\left(\phi^*(x_1)\phi(x_1)\varphi(x_1)\phi^*(x_2)\phi(x_2)\varphi(x_2)\right) = \dots + :\phi^*(x_1)\phi(x_1)\phi^*(x_2)\phi(x_2):\, \overbrace{\varphi(x_1)\varphi(x_2)}$$

any other term on the rhs of Wick's theorem will give zero when sandwiched between $\langle f|$ and $|i\rangle$:

$$\langle f|S-1|i\rangle = \frac{-g^2}{2}\Delta_F(x_1-x_2)\,\langle p_1'p_2'|:\phi^\dagger(x_1)\phi(x_1)\phi^\dagger(x_2)\phi(x_2)|p_1 p_2\rangle$$

$$\langle p_1'p_2'|:\phi^\dagger(x_1)\phi(x_1)\phi^\dagger(x_2)\phi(x_2)|p_1p_2\rangle = \langle p_1'p_2'|\phi^\dagger(x_1)\phi^\dagger(x_2)|0\rangle\langle 0|\phi(x_1)\phi(x_2)|p_1p_2\rangle$$

etc.