We have introduced real and complex scaler fields so far, let us repeat their definition here to have a coherent notation:

$$\varphi = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\vec p}}} \left(a_{\vec p}\, e^{-ipx} + a^+_{\vec p}\, e^{ipx} \right)$$

$$\phi = \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_{\vec p}}} \left(b_{\vec p}\, e^{-ipx} + c^+_{\vec p}\, e^{ipx} \right)$$

$$\phi^* = \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_{\vec p}}} \left(c_{\vec p}\, e^{-ipx} + b^+_{\vec p}\, e^{ipx} \right)$$

We have seen that when we make a calculation of an S-matrix element we have to pair/contract the fields in the interaction Lagrangian either with other fields in $\mathcal{L}_I$ or with creation/annihilation operators in the external states. If any creation/annihil. operators either in $\mathcal{L}_I$ or in the external states is not paired with anything else it will annihilate the vacuum and give zero. So, the problem becomes a problem of finding all possible ways to make these pairings, and then to write down what comes out of these pairings/contractions.

Let us remind ourselves that the contraction of two fields in $\mathcal{L}_I$ will give a Feynman propagator:

$$\overbrace{\varphi(x)\,\varphi(y)} = \Delta_F(x-y) = i\int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2-m^2+i\varepsilon}$$

$$\overbrace{\phi(x)\,\phi^+(y)} = \Delta_F(x-y)$$

and that the pairing of a field with the corresponding creation/ann. operator works as follows:

$$\varphi(x)\sqrt{2E_{\vec p}}\, a^+_{\vec p}|0\rangle = \int \frac{d^3k}{(2\pi)^3\sqrt{2E_{\vec k}}} \left(a_{\vec k}\, e^{-ikx} + a^+_{\vec k}\, e^{ikx} \right) \sqrt{2E_{\vec p}}\, a^+_{\vec p}|0\rangle$$

$$= \int \frac{d^3k}{(2\pi)^3 \sqrt{2E_{\vec{k}}}} \sqrt{2E_{\vec{p}}}\, e^{-ikx} (2\pi)^3 \delta^3(\vec{k}-\vec{p}) |0\rangle + ("a^\dagger_{\vec{k}} a^\dagger_{\vec{p}}")|0\rangle$$

$$= e^{-ipx}|0\rangle + "a^\dagger_{\vec{k}} a^\dagger_{\vec{p}}" |0\rangle$$

$$\langle 0|\varphi(x)|\vec{p}\rangle = e^{-ipx}$$

$\langle \vec{p}|\varphi(x)|0\rangle = e^{ipx}$ and similarly for ϕ:

$$\langle 0|\phi(x)|N(\vec{p})\rangle = e^{-ipx} \longleftrightarrow \langle N(\vec{p})|\phi^\dagger(x)|0\rangle = e^{ipx}$$

$$\langle 0|\phi^\dagger(x)|\bar{N}(\vec{p})\rangle = e^{-ipx} \longrightarrow \langle \bar{N}(\vec{p})|\phi(x)|0\rangle = e^{ipx}$$

We also remind ourselves that each $\mathcal{L}_I$ comes with an integral over x:

$\int d^4x$ which, since the only x-dependence is either in the propagators or in the exponential generated by the external states, and since the propagators expressed as inverse Fourier transform also have the x-dep. in the form of an exponential, will generate a δ-function with argument the sum of the momenta incoming in the vertex.

If there is only one $\mathcal{L}_I$, this δ-function imposes momentum conservation in the process. If there are two $\mathcal{L}_I$'s two δ-funct. are generated, but one makes the integral over momentum in the definition of the propagator as inverse Fourier transform trivial. In other words: momentum conservation is guaranteed at each vertex so that what remains is always a δ-funct. implying overall momentum conservation in the process. If we define the scattering amplitude as the coefficient of this δ-function:

$$\langle f|S-1|i\rangle = A_{fi} (2\pi)^4 \delta^4(p_f - p_i)$$

we can formulate the following rules for calculating A_{fi} starting from a number of diagrams drawn according to the following logic:

- draw different kind of lines for each particle, but distinguish particles and antiparticles from the direction of an arrow which symbolyzes flow of charge:

φ : - - - - - -

ϕ : $\longrightarrow$ or $\longleftarrow$

- for each particle in the external states draw such a line from an arbitrary different point: initial states on the left, final states on the right;
- for each $\mathcal{L}_I$ draw a vertex with as many lines coming out of the vertex as fields in $\mathcal{L}_I$; the line has to be of the type assigned to the particle;
- connect the lines in all possible, topologically different ways.

For each diagram write down a mathematical expression according to the following rules:

- for each vertex a factor $-ig$
- for each internal line a factor $\frac{i}{k^2-m^2+i\varepsilon}$ where k is the momentum following from momentum conservation at each vertex;
- if there are loops in the diagram, the momentum flowing inside the loop is unconstrained by momentum conservation: integrate over the loop momentum: $\int \frac{d^4k}{(2\pi)^4}$

Let's now have a look at the $\lambda\phi^4$ theory:

$$\mathcal{L}_I = -\frac{\lambda}{4!}\varphi^4 \quad , \quad \mathcal{H}_I = \frac{\lambda}{4!}\varphi^4$$

If we consider a scattering process like:

$$\varphi(\vec{p}_1)\varphi(\vec{p}_2) \to \varphi(\vec{p}_3)\varphi(\vec{p}_4)$$

we have to calculate

$$\langle \vec{p}_3, \vec{p}_4 | -i\int d^4x\, \mathcal{H}_I(x)\, | \vec{p}_1 \vec{p}_2 \rangle =$$

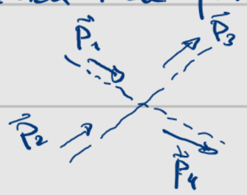
$$= -\frac{i\lambda}{4!}\int d^4x\, \langle 0 | a_{\vec{p}_3} a_{\vec{p}_4} \sqrt{2E_{\vec{p}_3}}\sqrt{2E_{\vec{p}_4}}\, \varphi^4(x)\, a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_2} | 0 \rangle$$

$$\varphi^4(x) = \int \frac{dk_1}{(2\pi)^3\sqrt{2E_{k_1}}} \cdots \frac{dk_4}{(2\pi)^3\sqrt{2E_{k_4}}} \left(a_{\vec{k}_1} e^{-ik_1x} + a^\dagger_{\vec{k}_1} e^{ik_1x}\right) \cdots \left(a_{\vec{k}_4} e^{-ik_4x} + a^\dagger_{\vec{k}_4} e^{ik_4x}\right)$$

In the end we need to count all possible ways we have to pair four a-ops. in $\mathcal{H}_I$ with the four ones in external states. This is combinatorics and gives a $4!$ which compensates the one in the denominator introduced for convenience. All the rest combines to give the δ-function of momentum conservation, and the scattering amplitude is just:

$$A(\varphi\varphi \to \varphi\varphi) = -i\lambda$$

This is also the Feynman rule for the vertex



$-i\lambda$

All other Feynman rules remain the same.

Mandelstam variables. The amplitude A_{fi} for a scattering process involving scalars is a Lorentz scalar. It is in principle a function of the four four-momenta $p_1, \ldots, p_4$, but can only depend on the scalar products which one can build out of them. There are two independent ones, which can be chosen as two of the Mandelstam variables:

$$s = (p_1+p_2)^2 = (p_3+p_4)^2$$

$$t = (p_1-p_3)^2 = (p_2-p_4)^2$$

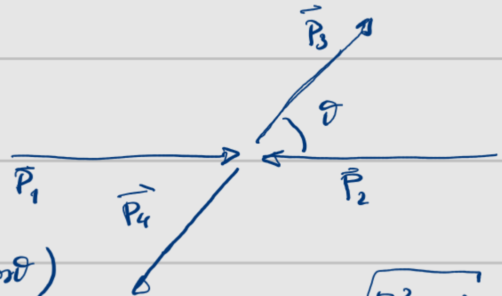
$$u = (p_1-p_4)^2 = (p_2-p_3)^2$$

These are not independent because:

$$s+t+u = 2m^2 + 2p_1\cdot p_2 + 2m^2 - 2p_1\cdot p_3 + 2m^2 - 2p_1\cdot p_4$$

$$= 6m^2 + 2p_1(\underbrace{p_2 - p_3 - p_4}_{-p_1}) = 4m^2$$

The relation to scattering angles depends on the reference frame, but it is worthwhile to work it out for the center-of-mass frame. In this case we have:



$$p_1 = (E, 0, 0, p) \;;\; p_2 = (E, 0, 0, -p)$$

$$p_3 = (E, 0, p\sin\vartheta, p\cos\vartheta) \;;\; p_4 = (E, 0, -p\sin\vartheta, -p\cos\vartheta)$$

$$p = \sqrt{E^2 - m^2}$$

$$s = 4E^2, \qquad t = (p_1-p_3)^2 = (0, 0, -p\sin\vartheta, p(1-\cos\vartheta))^2 = -p^2(\sin^2\vartheta + (1-\cos\vartheta)^2)$$

$$p^2 = E^2 - m^2 = \frac{s}{4} - m^2 \qquad = -p^2 2\cdot(1-\cos\vartheta) = -\frac{s}{2}\left(1 - \frac{4m^2}{s}\right)(1-\cos\vartheta)$$

$$s = 4E^2 \;;\; t = \frac{1}{2}(4m^2 - s)(1-\cos\vartheta) \;;\; u = \frac{1}{2}(4m^2-s)(1+\cos\vartheta)$$

$$t = \frac{s}{2}\sigma^2(s)(z-1) \qquad u = -\frac{s}{2}\sigma^2(s)(1+z)$$

Cross section and decay rates.

Consider an initial state consisting of a single particle, and a final state of two (or more) other particles. We want to calculate the transition probability, which is given by:

$$P = \frac{|\langle f|S|i\rangle|^2}{\langle f|f\rangle \langle i|i\rangle}$$

The denominator is necessary because our state are not properly normal.

$$\langle i|i\rangle = (2\pi)^3\, 2E_{\vec{p}_I}\, \delta^3(0) = 2E_{\vec{p}_I} V$$

where we have used:

$$\delta^3(\vec{p}) = \int \frac{d^3x}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \xrightarrow{\vec{p}\to 0} \frac{V}{(2\pi)^3}$$

For each particle in the final state we get a similar factor:

$$\langle f|f\rangle = \prod_i 2E_{\vec{p}_i}\cdot V \qquad p_F = \sum_i p_i$$

The matrix element

$$\langle f|S|i\rangle = (2\pi)^4 \delta^4(p_F - p_I)\, i\mathcal{A}_{fi}$$

contains a δ-function which we need to square. The second δ-function can be reinterpreted as a factor VT (including $(2\pi)^4$), so that we get:

(for $\vec{p}_I = 0$, $E_0 = m$)

$$P = \frac{|\mathcal{A}_{fi}|^2 (2\pi)^4 \delta^4(p_F - p_I)\cancel{V}T}{2m\cancel{V}} \cdot \prod_i \frac{1}{2E_{\vec{p}_i}V}$$

and dividing by T we obtain the transition probability per unit time, or decay rate. The total decay rate is obtained by summing over all possible final states or, for a given final state, over all possible momentum configurations. This is done by multiplying the probability by

$$\prod_i V \int \frac{d^3 p_i}{(2\pi)^3}$$

which gives the following density of final states:

$$d\Pi = (2\pi)^4 \delta^4(p_F - p_I) \prod_i \frac{d^3 p_i}{(2\pi)^3 2E_{\vec{p}_i}}$$

Note that the volume factors have all dropped out. If there are more final states possible we have to sum over them. The final decay rate is:

$$\Gamma = \frac{1}{2m} \sum_{\substack{\text{final}\\ \text{states}}} \int |A_{fi}|^2 d\Pi$$

Cross sections.

N = scattering events per unit time

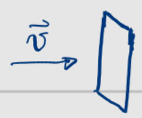
$= F \cdot \sigma$, F = flux per unit time and surface

σ = cross section

If we look at the number of scattered particles through a small cone $d\Omega$ around the direction identified by (ϑ, φ), then we have:

$$dN = F \cdot d\sigma$$

$$d\sigma = \frac{\text{diff. probability}}{\text{unit time} \times \text{unit flux}} = \frac{1}{F} \cdot \frac{1}{2E_1 2E_2 V} |A_{fi}|^2 \cdot d\Pi$$

What is the flux? $F =$ 

n. of particles going through the surface per unit time and surface. For a single-particle state

$$F = \frac{|\vec{v}|}{V}$$

In the CoM frame we have to replace $\vec{v}$ with the relative velocity:

$$F = \frac{|\vec{v}_1 - \vec{v}_2|}{V}$$

$$\Rightarrow \quad d\sigma = \frac{1}{4E_1E_2|\vec{v}_1-\vec{v}_2|}\,|A_{fi}|^2\, d\Pi$$

For example, for a 2→2 scattering process with identical particles we have:

$$\frac{1}{4E_1E_2|\vec{v}_1-\vec{v}_2|} = \frac{1}{4E^2\left|\frac{\vec{p}}{E}+\frac{\vec{p}}{E}\right|} = \frac{1}{8E^2\cdot\frac{|\vec{p}|}{E}} = \frac{1}{2s\,\sigma(s)}$$

$$\frac{|\vec{p}|}{E} = \frac{\sqrt{E^2-m^2}}{E} = \sqrt{1-\frac{m^2}{E^2}} = \underbrace{\sqrt{1-\frac{4m^2}{s}}}_{\sigma(s)}$$

$$d\Pi = \iint (2\pi)^4\,\delta^4(p_f-p_i)\,\frac{d^3p_3}{(2\pi)^3 2E_3}\,\frac{d^3p_4}{(2\pi)^3 2E_4} = \frac{1}{(2\pi)^2}\int\frac{d^3p_3}{2E_3}\cdot\delta\left((p_1+p_2-p_3)^2-m^2\right)$$

$$\frac{d^3p_4}{(2\pi)^3 2E_4} = \frac{d^4p_4}{(2\pi)^3}\,\delta(p_4^2-m^2)$$

$$(p_1+p_2-p_3)^2-m^2 = (2E-E_3)^2-\vec{p}_3^{\,2}-m^2 = 4E^2-4EE_3+\underbrace{E_3^2-\vec{p}_3^{\,2}-m^2}_{=0}$$

$$= 4E(E-E_3)$$

$$\frac{1}{(2\pi)^2}\int\frac{d^3p_3}{2E_3}\,\delta\left(4E(E-E_3)\right) = \frac{1}{(2\pi)^2}\int\frac{dp_3\cdot p_3^2\,d\Omega}{2E_3}\,\delta\left(4E(E-E_3)\right) =$$

$$dp_3\cdot p_3 = dE_3\cdot E_3$$

$$= \frac{1}{(2\pi)^2}\int\frac{d\Omega\; p\cancel{E}}{2\cancel{E}}\,\frac{1}{4E} =$$

$$= \frac{\sigma(s)}{8}\,\frac{1}{(2\pi)^2}\int d\Omega$$

Putting everything back together:

$$\frac{d\sigma}{d\Omega} = \frac{1}{2s\cancel{\sigma(s)}}\cdot\frac{\cancel{\sigma(s)}}{8}\,\frac{1}{(2\pi)^2}\,|A_{fi}|^2 = \frac{1}{16s}\,\frac{|A_{fi}|^2}{4\pi^2}$$

So, for a 2→2 scattering there is a general $\frac{1}{s}$ behavior, related to the dimensions of the cross section (surface $= [E]^{-2}$). This multiplies the amplitude squared, which is dimensionless.

Discuss the qualitative behavior of cross sections resulting from some of the amplitudes we have calculated.

$s \geq 4M^2$; $s+t+u = 4M^2 \Rightarrow t,u \leq 0$

$$A(NN \to NN) = g^2 \left(\frac{1}{t-m^2} + \frac{1}{u-m^2} \right) \qquad [g]=[E]$$

$$= g^2 \left(\frac{1}{-\frac{s}{2}\sigma^2(1-z)-m^2} + \frac{1}{-\frac{s}{2}\sigma^2(1+z)-m^2} \right)$$

$$= g^2 \frac{-s\sigma^2-2m^2}{\left(\frac{s}{2}\sigma^2+m^2\right)^2-\left(\frac{s}{2}\sigma^2 z\right)^2} = \frac{-s+4m^2-2m^2}{\frac{(s-2m^2)^2}{4}-\frac{(s-4m^2)^2 z^2}{4}} =$$

$$\boxed{\frac{s}{2}\sigma^2+m^2 = \frac{s}{2}-\cancel{2}m^2+\cancel{m^2} = \frac{s-2m^2}{2}}$$

$$= \frac{-4}{s-2m^2} \cdot \frac{1}{1-\left(\frac{s-4m^2}{s-2m^2}\right)^2 z^2}$$

$$A(N\bar{N} \to N\bar{N}) = g^2 \left(\frac{1}{s-m^2+i\varepsilon} + \frac{1}{t-m^2} \right)$$

$s \geq 4M^2$; if $m^2 < 4M^2$ there is no divergence $\Rightarrow$ no need for $i\varepsilon$

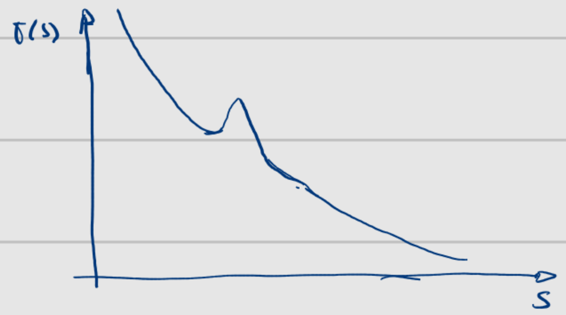
if $m^2 > 4M^2$ $\quad s-m^2$ can be zero; but then $\varphi \to N\bar{N}$ and

$\Gamma(\varphi \to N\bar{N}) \neq 0$

$m^2 \to (m-i\Gamma)^2 = m^2-\Gamma^2-2im\Gamma$

the particle is unstable and decays.

In this case the observed cross section shows a bump in the distribution in s:



$$A(\varphi\varphi \to \varphi\varphi) = -i\lambda \quad \text{in} \quad \lambda\varphi^4$$

$\Rightarrow |A_{fi}|^2 = \lambda^2$ isotropic scattering, and the cross section drops like $\frac{1}{s}$.

Yukawa potential.

Inhomogeneous relativistic wave eq.

$$(\Box + m^2)\,\phi(x) = f(x) \quad = \text{source}$$

What field configuration satisfies this eq. if we take as source a δ-function independent of time?

$$(-\Delta + m^2)\,\phi(\vec{x}) = \delta(\vec{x})$$

$\phi(\vec{x})$ as inverse Fourier transform:

$$\phi(\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}\, \tilde{\phi}(\vec{k})$$

Solution: $\tilde{\phi}(\vec{k}) = \dfrac{1}{\vec{k}^2 + m^2}$. Let's calculate $\phi(\vec{x})$ explicitly, by solving the integral in polar coordinates:

$$\phi(\vec{x}) = \frac{1}{(2\pi)^2}\int_0^\infty dk\, \frac{k^2}{k^2+m^2}\,\frac{2\sin kr}{kr} = \frac{1}{(2\pi)^2 r}\int_{-\infty}^{\infty} dk\, \frac{k\sin kr}{k^2+m^2}$$

$\sin kr = \mathrm{Re}\left(\dfrac{e^{ikr}}{i}\right)$

$$= \frac{1}{2\pi r}\,\mathrm{Re}\left[\int_{-\infty}^{\infty} \frac{dk}{2\pi i}\,\frac{k e^{ikr}}{k^2+m^2}\right]$$

Poles at $k = \pm im$. Closing the contour for $k \to \mathrm{Re}(k) + i\infty$ gives a vanishing contribution from the arc at infinity.

$$\Rightarrow \phi(\vec{x}) = \frac{1}{2\pi r}\,\frac{im\,e^{-mr}}{2\cdot im} = \frac{1}{4\pi r}\,e^{-mr}$$

Can we relate this result to a calculation in QFT?

Let us look again at the $NN \to NN$ scattering amplitude and take the nonrelativistic limit of the amplitude.

$$i\mathcal{A}(NN \to NN) = ig^2\left[\frac{1}{(\vec{p}_1 - \vec{p}_3)^2 - m^2} + \frac{1}{(\vec{p}_1 + \vec{p}_3)^2 - m^2}\right]$$

the second term would be $(\vec{p}_1 - \vec{p}_4)^2 = (\vec{p}_1 + \vec{p}_3)^2$

Can we interpret this in terms of a potential in a quantum mech. setting? Yes, the same amplitude would have to be obtained as matrix element of a potential $V(\vec{r})$ as follows:

$$\langle \vec{p}_3 | V(\vec{r}) | \vec{p}_1 \rangle = -i \int d^3r \, V(\vec{r}) \, e^{-i(\vec{p}_1-\vec{p}_3)\vec{r}}$$

This has to be equal to the amplitude we calculated (modulo a factor $(2M)^2$ due to the difference between rel. and non rel. normalization of states), so we have

$$\int d^3r \, V(\vec{r}) \, e^{-i(\vec{p}_1-\vec{p}_3)\cdot\vec{r}} = \frac{-\lambda^2}{(\vec{p}_1-\vec{p}_3)^2 - m^2} \qquad \lambda = \frac{g}{2M}$$

By simply inverting the Fourier transform we get

$$V(\vec{r}) = -\lambda^2 \int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p}\cdot\vec{r}}}{\vec{p}^{\,2}+m^2}$$

which is just the same integral we already solved above.

$$\Rightarrow \quad V(\vec{r}) = \frac{-\lambda^2}{4\pi r} e^{-mr} \quad = \text{Yukawa potential.}$$

(Nobel prize 1949 for predicting the existence of the π-meson).