Having completed the construction of a QFT for a scalar field, we now want to extend it to other fields transforming in more complicated ways under Lorentz transformations.

In particular: for $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$$

You know already other kinds of Lorentz covariant fields, like the 4-potential of em. They transform according to

$$A^\mu(x) \to \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x).$$

The latter is a particular case of a general form:

$$\phi^a(x) \to D(\Lambda)^a{}_b \phi^b(\Lambda^{-1}x) \quad \text{where } a,b=1,\dots,n$$

where D must satisfy

$$D(\Lambda_1)D(\Lambda_2) = D(\Lambda_1\Lambda_2) \qquad \forall \Lambda_{1,2} \in L \text{ (Lorentz group)}$$

as well as $D(\mathbb{1}) = 1$ and $D(\Lambda^{-1}) = D^{-1}(\Lambda)$.

In order to find these representations one can consider infinitesimal transformations and thereby concentrate on the algebra and their reps.

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$$

with $\omega^\mu{}_\nu$ infinitesimal parameters. The condition:

$\Lambda^\mu{}_\sigma \Lambda^\nu{}_\rho g^{\sigma\rho} = g^{\mu\nu}$ implies:

$$(\delta^\mu{}_\sigma + \omega^\mu{}_\sigma)(\delta^\nu{}_\rho + \omega^\nu{}_\rho)g^{\sigma\rho} = g^{\mu\nu} + \omega^{\mu\nu} + \omega^{\nu\mu} + \mathcal{O}(\omega^2) = g^{\mu\nu}$$

$$\Rightarrow \omega^{\mu\nu} + \omega^{\nu\mu} = 0$$

$\omega^{\mu\nu}$ is an antisymmetric 4×4 matrix

We now introduce a basis for 4x4 antisymmetric matrices. This basis has 6 elements, like the number of independent parameters in 4x4 As matrices. We call them $(M^A)^{\mu\nu}$, $A = 1, \dots, 6$, but in fact it's more convenient to write

$(M^{\rho\sigma})^{\mu\nu}$, thereby meaning that $M^{\rho\sigma}$ are also antisymm. in the indices:

$M^{\rho\rho} = 0$; $M^{\rho\sigma} = -M^{\sigma\rho}$, which makes only 6 of them independent.

Such a basis can be written explicitly as:

$$(M^{\rho\sigma})^{\mu\nu} = g^{\rho\mu} g^{\sigma\nu} - g^{\sigma\mu} g^{\rho\nu}$$

Remember that when acting on any other object one of the index of our matrices will have to be lowered, which will make a Kronecker-δ out of the metric:

$$(M^{\rho\sigma})^{\mu}{}_{\nu} = g^{\rho\mu} \delta^{\sigma}{}_{\nu} - g^{\sigma\mu} \delta^{\rho}{}_{\nu}$$

Examples:

$$M^{01} = g^{0\mu} \delta^{1}{}_{\nu} - g^{1\mu} \delta^{0}{}_{\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

= boost in the x direction

$$M^{12} = g^{1\mu} \delta^{2}{}_{\nu} - g^{2\mu} \delta^{1}{}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

= rotation around the z-axis.

Any $\omega^{\mu}{}_{\nu}$ can now be written in this basis:

$$\omega^{\mu}{}_{\nu} = \frac{1}{2} \Omega_{\rho\sigma} (M^{\rho\sigma})^{\mu}{}_{\nu}$$

$M^{\rho\sigma}$ are called the generators of the Lorentz transf.

Commutation relations among them (= Lie algebra):

$$[M^{\rho\sigma}, M^{\tau\nu}] = g^{\sigma\tau} M^{\rho\nu} - g^{\rho\tau} M^{\sigma\nu} + g^{\rho\nu} M^{\sigma\tau} - g^{\sigma\nu} M^{\rho\tau}.$$

$$\Lambda = \exp\left(\frac{1}{2}\Omega_{\rho\sigma} M^{\rho\sigma}\right)$$

Spinor representation

We start from the Clifford algebra:

$$\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu} \mathbb{1} \qquad \gamma^\mu, \mu = 0, \dots, 3$$

$$\Rightarrow \quad \gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu \quad \text{if } \mu \neq \nu \; ; \; (\gamma^0)^2 = 1 \; ; \; (\gamma^i)^2 = -1$$

If one tries to satisfy these conditions with matrices, one quickly realizes that the first solution can be obtained with 4x4 matrices.

For example through the following explicit solution:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; ; \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad , \sigma^i \text{ the Pauli matrices}$$

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij}$$

Consider the commutator of two γ^μ's:

$$S^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu] = \frac{1}{2}\left(\gamma^\mu\gamma^\nu - \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\right) = \frac{1}{2}\left(\gamma^\mu\gamma^\nu - g^{\mu\nu}\right)$$

$$= \frac{1}{2}\gamma^\mu\gamma^\nu \text{ if } \mu \neq \nu$$

Properties of the $S^{\mu\nu}$:

1. $[S^{\mu\nu}, \gamma^\rho] = \gamma^\mu g^{\nu\rho} - \gamma^\nu g^{\rho\mu}$

Proof: for $\mu = \nu$ it is trivial.

$$[S^{\mu\nu}, \gamma^\rho] = \frac{1}{2}[\gamma^\mu\gamma^\nu, \gamma^\rho] = \frac{1}{2}\gamma^\mu\gamma^\nu\gamma^\rho - \frac{1}{2}\gamma^\rho\gamma^\mu\gamma^\nu =$$

$$= \frac{1}{2}\gamma^\mu\{\gamma^\nu, \gamma^\rho\} - \cancel{\frac{1}{2}\gamma^\mu\gamma^\rho\gamma^\nu} - \frac{1}{2}\{\gamma^\rho, \gamma^\mu\}\gamma^\nu + \cancel{\frac{1}{2}\gamma^\mu\gamma^\rho\gamma^\nu}$$

$$= \gamma^\mu g^{\nu\rho} - \gamma^\nu g^{\mu\rho} \qquad \checkmark$$

2. The $S^{\mu\nu}$ form a representation of the Lorentz algebra.

Proof:

$$[S^{\mu\nu}, S^{\rho\sigma}] \overset{\rho\neq\sigma}{=} \frac{1}{2}[S^{\mu\nu}, \gamma^\rho\gamma^\sigma] = \frac{1}{2}[S^{\mu\nu}, \gamma^\rho]\gamma^\sigma + \frac{1}{2}\gamma^\rho[S^{\mu\nu}, \gamma^\sigma]$$

$$= \frac{1}{2}\gamma^\mu\gamma^\sigma g^{\nu\rho} - \frac{1}{2}\gamma^\nu\gamma^\sigma g^{\mu\rho} + \frac{1}{2}\gamma^\rho\gamma^\mu g^{\nu\sigma} - \frac{1}{2}\gamma^\rho\gamma^\nu g^{\mu\sigma}$$

$$= S^{\mu\sigma} g^{\nu\rho} - S^{\nu\sigma} g^{\mu\rho} + S^{\rho\mu} g^{\nu\sigma} - S^{\rho\nu} g^{\mu\sigma} \quad \checkmark$$

Spinors The fields on which $S^{\mu\nu}$ act on are called Dirac spinors.

$$\psi^\alpha(x) \to S(\Lambda)^\alpha{}_\beta\, \psi^\beta(\Lambda^{-1}x)$$

where $\Lambda = \exp\left[\frac{1}{2}\Omega_{\rho\sigma} M^{\rho\sigma}\right]$

$$S[\Lambda] = \exp\left[\frac{1}{2}\Omega_{\rho\sigma} S^{\rho\sigma}\right]$$

The $S[\Lambda]$ are 4×4 matrices which satisfy the same algebraic properties of Λ, so we could have simply rediscovered the same thing and given them a new name. Is this so?

Let us consider rotations and see what form $S[R]$ takes in this case:

$$S^{ij} = \frac{1}{2}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} \sigma^i\sigma^j & 0 \\ 0 & \sigma^i\sigma^j \end{pmatrix}$$

$$\sigma^i\sigma^j = \frac{1}{2}\{\sigma^i, \sigma^j\} + \frac{1}{2}[\sigma^i, \sigma^j] \overset{i\neq j}{=} i\varepsilon_{ijk}\sigma^k$$

$$\Rightarrow S^{ij} = -\frac{i}{2}\varepsilon_{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

and, with $\Omega_{ij} = -\varepsilon_{ijk}\varphi_k$

$$\Rightarrow S[\Lambda] = \exp\left[-\frac{1}{2}\varepsilon_{ijk}\varphi_k \cdot \left(\frac{-i}{2}\right)\varepsilon^{ijl}\begin{pmatrix} \sigma_l & 0 \\ 0 & \sigma_l \end{pmatrix}\right]$$

$$= \exp\left[\frac{i}{4}\, \varepsilon_{ijk}\varepsilon^{ijl}\, \varphi_k \begin{pmatrix} \sigma_l & 0 \\ 0 & \sigma_l \end{pmatrix}\right] =$$

$$\left(\varepsilon_{ijk} \cdot \varepsilon^{ijl} = 2\delta^l{}_k \right) \to = \exp\left[\frac{i}{2}\varphi_l \begin{pmatrix} \sigma_l & 0 \\ 0 & \sigma_l \end{pmatrix}\right]$$

$$= \begin{bmatrix} \exp\left(i\frac{\vec{\varphi}\cdot\vec{\sigma}}{2}\right) & 0 \\ 0 & \exp\left(i\frac{\vec{\varphi}\cdot\vec{\sigma}}{2}\right) \end{bmatrix}$$

If we make a $\varphi = 2\pi$ rotation about the z-axis, we need to set $\vec{\varphi} = (0,0,2\pi)$, which yields:

$$S[\Lambda] = \begin{pmatrix} e^{i\pi\sigma_3} & 0 \\ 0 & e^{i\pi\sigma_3} \end{pmatrix} = -\mathbb{1}$$

$$e^{i\varphi\sigma_3} = \mathbb{1}\cos\varphi + i\sigma_3 \sin\varphi$$

$$\Rightarrow \quad \psi^\alpha(x) \xrightarrow{\varphi_3 = 2\pi} -\psi^\alpha(x)$$

If we instead evaluate Λ we get:

$$\Lambda = \exp\left(\frac{1}{2}\varepsilon_{ij3}\varphi_3 \left(g^{i\mu}g^{j\nu} - g^{j\mu}g^{i\nu}\right)\right) =$$

$$\exp\left[\varepsilon_{ij3}\varphi_3 \left(g^{i\mu}g^{j\nu}\right)\right] = \exp\left[\varphi_3\left(g^{1\mu}g^{2\nu} - g^{2\mu}g^{1\nu}\right)\right]$$

$$= \exp\underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \varphi_3 & 0 \\ 0 & -\varphi_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{\Phi_3}$$

$$\Phi_3 = i\varphi_3\hat{\sigma}_2 \qquad \hat{\sigma}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \hat{\mathbb{1}}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathbb{1} & & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\exp(i\varphi_3\hat{\sigma}_2) = \mathbb{1} + (\cos\varphi_3 - 1)\,\hat{\mathbb{1}}_2 + i\sin\varphi_3\,\hat{\sigma}_2 =$$

$$= \mathbb{1} \quad \text{for } \varphi_3 = 2\pi$$

For $\Omega_{ij} = \varepsilon_{ij3}\cdot 2\pi$ $S[\Lambda] = -\mathbb{1}$, whereas $\Lambda = \mathbb{1}$.

__Boosts__ How do boosts look like through the S-rep?

$$S^{0i} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$$

For $\Omega_{i0} = -\Omega_{0i} = \chi_i$ we have

$$S[\Lambda] = \exp\left(\frac{1}{2}\Omega_{\mu\nu}S^{\mu\nu}\right) = \exp\left(\frac{1}{2}\chi_i\begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}\right)$$

$$= \begin{pmatrix} e^{\vec{\chi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{-\vec{\chi}\cdot\vec{\sigma}/2} \end{pmatrix}$$

Here it is important to remark that while the representation of rotations are unitary matrices, those of boosts are not. This is because it can be proven that there are no finite dimensional unitary representations of the Lorentz group.

__Constructing an action.__

Let's try to build Lorentz scalars out of the Dirac spinors. The first naive attempt could be to multiply $\psi^\dagger$ with ψ, however:

$$\psi(x) \to S[\Lambda]\psi(\Lambda^{-1}x) \quad \text{which implies}$$

$$\psi^\dagger(x) \to \psi^\dagger(\Lambda^{-1}x)S^\dagger[\Lambda]$$

$$\Rightarrow \psi^\dagger(x)\psi(x) \to \psi^\dagger(\Lambda^{-1}x)S^\dagger[\Lambda]S[\Lambda]\psi(\Lambda^{-1}x)$$

but since $S[\Lambda]$ is not unitary, this is not a scalar.

The solution is close, however, because we have seen that:

$$\gamma^{0\dagger} = \gamma^0 \text{ and } \gamma^{i\dagger} = -\gamma^i, \text{ while at the same time}$$

$$\gamma^i\gamma^0 = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} \to \gamma^0\gamma^i\gamma^0 = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix} = -\gamma^i = \gamma^{i\dagger}$$

This immediately generalizes to $\gamma^0 \gamma^\mu \gamma^0 = \gamma^{\mu\dagger}$

$$\Rightarrow \quad (S^{\mu\nu})^\dagger = \frac{1}{4}\left[\gamma^{\nu\dagger}, \gamma^{\mu\dagger}\right] = \frac{1}{4}\left[\gamma^0\gamma^\nu\gamma^0, \gamma^0\gamma^\mu\gamma^0\right] =$$
$$= \frac{1}{4}\gamma^0\left[\gamma^\nu, \gamma^\mu\right]\gamma^0$$
$$= -\gamma^0 S^{\mu\nu}\gamma^0$$

which implies
$$S[\Lambda]^\dagger = \exp\left[\frac{1}{2}\Omega_{\rho\sigma}(S^{\rho\sigma})^\dagger\right] =$$
$$= \exp\left[-\frac{1}{2}\Omega_{\rho\sigma}\,\gamma^0 S^{\rho\sigma}\gamma^0\right] = \gamma^0 \exp\left[-\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right]\gamma^0$$
$$= \gamma^0 S[\Lambda]^{-1}\gamma^0$$

$\Rightarrow$ $\bar{\psi} \equiv \psi^\dagger\gamma^0$ is what we need to multiply ψ with if we want to build a Lorentz scalar:

$$\bar{\psi}(x)\psi(x) \longrightarrow \psi^\dagger(\Lambda^{-1}x)\, S^\dagger[\Lambda]\gamma^0 S[\Lambda]\psi(\Lambda^{-1}x) =$$
$$= \psi^\dagger(\Lambda^{-1}x)\gamma^0 \cdot \gamma^0 S^\dagger[\Lambda]\gamma^0 S[\Lambda]\psi(\Lambda^{-1}x) =$$
$$= \bar{\psi}(\Lambda^{-1}x)\, S^{-1}[\Lambda]S[\Lambda]\psi(\Lambda^{-1}x) = \bar{\psi}(\Lambda^{-1}x)\psi(\Lambda^{-1}x) \quad \checkmark$$

Analogously, we can also show that $\bar{\psi}\gamma^\mu\psi$ transforms like a 4-vector, namely:
$$\bar{\psi}\gamma^\mu\psi \longrightarrow \Lambda^\mu{}_\nu\, \bar{\psi}(\Lambda^{-1}x)\gamma^\nu\psi(\Lambda^{-1}x)$$

Let's do it: $\bar{\psi}\gamma^\mu\psi \longrightarrow \bar{\psi}\, S[\Lambda]^{-1}\gamma^\mu S[\Lambda]\psi$, so we need to show that
$$S[\Lambda]^{-1}\gamma^\mu S[\Lambda] = \Lambda^\mu{}_\nu\gamma^\nu$$

Let us consider infinitesimal transf.:

$$\Lambda = \exp\left[\frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}\right] \simeq 1 + \frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma} + \dots$$
$$S[\Lambda] = \exp\left[\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right] \simeq 1 + \frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma} + \dots$$

$$\Rightarrow \quad S[\Lambda]^{-1}\gamma^\mu S[\Lambda] = \gamma^\mu - \frac{1}{2}\Omega_{\rho\sigma}\left[S^{\rho\sigma}, \gamma^\mu\right] + O(\Omega^2) \overset{?}{=} \overbrace{\gamma^\mu + \frac{1}{2}\Omega_{\rho\sigma}(M^{\rho\sigma})^\mu{}_\nu\gamma^\nu}^{\Lambda^\mu{}_\nu\gamma^\nu} + O(\Omega^2)$$

But we have shown that $\left[S^{\rho\sigma}, \gamma^\mu\right] = \gamma^\rho g^{\sigma\mu} - \gamma^\sigma g^{\mu\rho}$

and since $\quad (M^{\rho\sigma})^{\mu}{}_{\nu}\gamma^{\nu} = \left(g^{\rho\mu}\delta^{\sigma}_{\nu} - g^{\sigma\mu}\delta^{\rho}_{\nu}\right)\gamma^{\nu} = g^{\rho\mu}\gamma^{\sigma} - g^{\sigma\mu}\gamma^{\rho}$ we get:

$S[\Lambda]^{-1}\gamma^{\mu}S[\Lambda] = \gamma^{\mu} + \frac{1}{2}\Omega_{\rho\sigma}(\gamma^{\sigma}g^{\mu\rho} - \gamma^{\rho}g^{\mu\sigma}) + \mathcal{O}(\Omega^2)$; $\Lambda^{\mu}{}_{\nu}\gamma^{\nu} = \gamma^{\mu} + \frac{1}{2}\Omega_{\rho\sigma}\left(\gamma^{\sigma}g^{\mu\rho} - \gamma^{\rho}g^{\mu\sigma}\right) + \mathcal{O}(\Omega^2)$

$\Longrightarrow$ the two sides of the equation are identical.

Similarly, we can prove that $\bar{\psi}\gamma^{\mu}\gamma^{\nu}\psi$ behaves like a Lorentz tensor. This provides the basis for constructing an action, which must be a Lorentz scalar, if we want a Lorentz-invariant theory. The simplest action which contains spacetime derivatives that we can build is:

$$S_D = \int d^4x\, \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi$$

The field equation derived from this action is

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0 \qquad \text{Dirac's equation.}$$

The presence of the γ-matrices mixes up the 4 components of the spinor ψ. However, each of these components also satisfies the KG equation as we can easily show:

[first a new notation if a^{μ} is a four-vector, $\not{a} = \gamma^{\mu}a_{\mu}$]

$$\Rightarrow (i\not{\partial} + m)(i\not{\partial} - m)\psi = -(\gamma^{\mu}\gamma^{\nu}\partial_{\mu}\partial_{\nu} + m^2)\psi = 0$$

$$\gamma^{\mu}\gamma^{\nu}\partial_{\mu}\partial_{\nu} = \frac{1}{2}\{\gamma^{\mu},\gamma^{\nu}\}\partial_{\mu}\partial_{\nu} = g^{\mu\nu}\partial_{\mu}\partial_{\nu} = \Box$$

$$\Rightarrow -(\Box + m^2)\psi = 0$$

<u>Chiral spinors.</u>

$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$; $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$ is the so-called chiral representation of γ-matrices.

In this representation both the spinor rotation as well as boost transformations are block diagonal:

$$S[\Lambda_{rot}] = \begin{pmatrix} e^{i\vec{\varphi}\vec{\sigma}/2} & 0 \\ 0 & e^{-i\vec{\varphi}\vec{\sigma}/2} \end{pmatrix} \; ; \; S[\Lambda_{boost}] = \begin{pmatrix} e^{\vec{\chi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{-\vec{\chi}\cdot\vec{\sigma}/2} \end{pmatrix}$$

$\Rightarrow$ the representation is reducible. Let us split the Dirac spinors into two two-component spinors:

$$\psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$$ $u_\pm$ are called Weyl spinors,

which transform:

$$u_\pm \to e^{i\vec{\varphi}\cdot\vec{\sigma}/2} \cdot u_\pm$$

$$u_\pm \to e^{\pm\vec{\chi}\cdot\vec{\varphi}/2} u_\pm \, .$$

If we write the Dirac equation for the Weyl spinors we get:

$$\bar{\psi}(i\partial\!\!\!/ - m)\psi = (u_+^\dagger \; u_-^\dagger) \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} (i\gamma^\mu \partial_\mu - m) \cdot \begin{pmatrix} u_+ \\ u_- \end{pmatrix} =$$

Let's now introduce the following 4-vectors: $\sigma_\mu = (\mathbb{1}, \sigma^i)$; $\bar{\sigma}_\mu = (\mathbb{1}, -\sigma^i)$.

and write: $\gamma^\mu \partial_\mu = \begin{pmatrix} 0 & \sigma^\mu \partial_\mu \\ \bar{\sigma}^\mu \partial_\mu & 0 \end{pmatrix}$; $(i\partial\!\!\!/ - m) = \begin{pmatrix} m & i\sigma^\mu \partial_\mu \\ i\bar{\sigma}^\mu \partial_\mu & m \end{pmatrix}$

and finally $\gamma_0(i\partial\!\!\!/ - m) = \begin{pmatrix} i\bar{\sigma}^\mu \partial_\mu & m \\ m & i\sigma^\mu \partial_\mu \end{pmatrix}$

$$\Rightarrow \bar{\psi}(i\partial\!\!\!/ - m)\psi = u_-^\dagger i\sigma_\mu \partial^\mu u_- + u_+^\dagger i\bar{\sigma}_\mu \partial^\mu u_+ - m(u_-^\dagger u_+ + u_+^\dagger u_-)$$

If the particle has mass the equations for the two Weyl fermions are coupled and there is no real advantage in introducing them. But if $m=0$, then the two eqs. decouple and become:

$$i\bar{\sigma}^\mu \partial_\mu u_+ = 0 \; ; \; i\sigma^\mu \partial_\mu u_- = 0$$

$S[\Lambda]$ is block-diagonal only in the chiral rep., but it would be useful to define Weyl fermions in a way which is independent of the repr.

To reach this goal we introduce the γ_5 matrix:

$$\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$$

which satisfies: $\{\gamma^5, \gamma^\mu\} = 0$, ${\gamma^5}^2 = \mathbb{1}$

$$P_\pm = \frac{1}{2}(1 \pm \gamma_5)$$

are projection operators:

$$P_+^2 = \frac{1}{4}(1 + 2\gamma_5 + \gamma_5^2) = \frac{1}{2}(1+\gamma_5) = P_+$$

$$P_-^2 = P_-$$

Moreover, in the chiral representation we have:

$$\gamma_5 = -i \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} =$$

$$-i \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \begin{pmatrix} -\sigma_2\sigma_3 & 0 \\ 0 & -\sigma_2\sigma_3 \end{pmatrix} = -i \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix}$$

$$= - \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

$$\gamma_5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

$$P_+ = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \; ; \; P_- = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

$\Rightarrow \psi_\pm = P_\pm \psi$; this definition is independent of the representation of the γ-matrices.

Parity and Time Reversal.

Discrete spacetime transformations:

T: $x^0 \to -x^0$; $x^i \to x^i$

P: $x^0 \to x^0$; $x^i \to -x^i$

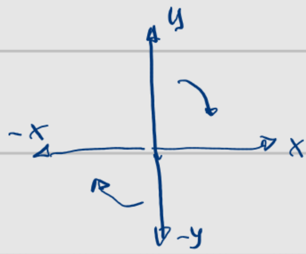
How do the spinors transform under these discrete transf.?

We will discuss time reversal at some later time.

Consider parity: if we apply a parity transf., rotations will not change their form, but boosts will change sign:

since $u_\pm \xrightarrow{rot} e^{i\vec{\varphi}\cdot\vec{\sigma}/2} u_\pm$ and $u_\pm \xrightarrow{boost} e^{\pm\vec{\chi}\cdot\vec{\sigma}/2} u_\pm$

consistency requires that $P: u_\pm \to u_\mp$



$$P: \psi_\pm(\vec{x},t) \to \psi_\mp(-\vec{x},t),$$

which can also be written as:

$$P: \psi(\vec{x},t) \to \gamma^0\psi(-\vec{x},t)$$

which also respects $P^2=1$

$$P_\pm\psi \to \gamma_0 P_\pm \gamma_0 \psi = \frac{\gamma_0}{2}(1\pm\gamma_5)\gamma_0\psi = \frac{1}{2}\gamma_0^2(1\mp\gamma_5)\psi = P_\mp\psi$$

We remark also that if $\psi(x)$ satisfies the Dirac equation:

$(i\partial\!\!\!/ - m)\psi = 0$, a parity transformation on ψ leads to:

$$(i\partial\!\!\!/ - m)\gamma_0\psi(-\vec{x},t) = (i\gamma^0\partial_0 - i\vec{\gamma}\cdot\vec{\partial} - m)\gamma_0\psi(-\vec{x},t) =$$

$$= \gamma_0(i\gamma^0\partial_0 + i\vec{\gamma}\cdot\vec{\partial} - m)\psi(-\vec{x},t) = \quad (-\vec{x} = \vec{y})$$

$$= \gamma_0(i\gamma^0\partial_0 - i\vec{\gamma}\cdot\vec{\partial}_y - m)\psi(\vec{y},t) = \gamma_0(i\partial\!\!\!/ - m)\psi = 0$$

<u>Parity transformations of bilinears</u>:

$$P: \bar{\psi}(x)\psi(x) \longrightarrow \psi^\dagger(-\vec{x},t)\gamma^0\gamma^0\gamma^0\psi(-\vec{x},t) = \underset{\text{scalar}}{\underset{\parallel}{\bar{\psi}(-\vec{x},t)\psi(-\vec{x},t)}}$$

$$P: \bar{\psi}\gamma^\mu\psi(x) \longrightarrow \begin{cases} \mu=0: & \psi^\dagger\gamma^0\gamma^0\gamma^0\gamma^0\psi(-\vec{x},t) = \bar{\psi}\gamma^0\psi(-\vec{x},t) \\ \mu=i: & \psi^\dagger\gamma^0\gamma^0\gamma^i\gamma^0\psi(-\vec{x},t) = -\bar{\psi}\gamma^i\psi(-\vec{x},t) \end{cases}$$

so $\bar{\psi}\gamma^\mu\psi$ transforms like a vector. But having introduced γ_5 we can form two more bilinears:

$\bar{\psi}\gamma_5\psi$ and $\bar{\psi}\gamma^\mu\gamma_5\psi$, for which it is easy to show that

$$P: \bar{\psi}\gamma_5\psi \to -\bar{\psi}\gamma_5\psi(-\vec{x},t)$$

$$P: \bar{\psi}\gamma^\mu\gamma_5\psi \to \begin{cases} -\bar{\psi}\gamma^0\gamma_5\psi(-\vec{x},t) & \mu=0 \\ \bar{\psi}\gamma^i\gamma_5\psi(-\vec{x},t) & \mu=i \end{cases}$$

So, all in all we have the following bilinears: $S: \bar{\psi}\psi$; $P: \bar{\psi}\gamma_5\psi$; $V: \bar{\psi}\gamma_\mu\psi$; $A: \bar{\psi}\gamma_\mu\gamma_5\psi$
$T: \bar{\psi}S_{\mu\nu}\psi$

Majorana fermions

The Majorana basis for the γ-matrices reads:

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} ; \gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix} ; \gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix} ; \gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}$$

All matrices are purely imaginary, so that $S^{\mu\nu}$ are purely real, which is also true for $S[\Lambda]$. In such a basis we can impose the reality condition for the spinors:

$$\psi = \psi^* \quad = \text{Majorana spinors.}$$

because this will not be upset by a Lorentz transformation.

But again, one property which is shown explicitly in one basis has to be formulated in general terms for a generic basis, which is only required to satisfy $\gamma^{0\dagger} = \gamma^0$ and $\gamma^{i\dagger} = -\gamma^i$.

We then introduce the charge conjugation transformation:

$$\psi^{(c)} = C\psi^*$$

where C is a 4×4 matrix which must satisfy:

$$C^\dagger C = 1 \; ; \; C^\dagger\gamma^\mu C = -\gamma^{\mu *}.$$

$$C\gamma^{\mu *} = -\gamma^\mu C$$

How do $\psi^{(c)}$ transform?

$$\psi^{(c)} \to C\, S[\Lambda]^* \psi^* = S[\Lambda] C \psi^* = S[\Lambda]\psi^{(c)}$$

$$C[\gamma^{\mu *}, \gamma^{\nu *}] = C(\gamma^{\mu *}\gamma^{\nu *} - \gamma^{\nu *}\gamma^{\mu *}) = -\gamma^{\mu} C \gamma^{\nu *} + \gamma^{\nu} C \gamma^{\mu *}$$

$$= \gamma^{\mu}\gamma^{\nu} C - \gamma^{\nu}\gamma^{\mu} C = [\gamma^{\mu}, \gamma^{\nu}] C$$

$\psi^{(c)}$ also satisfies Dirac's equation (if ψ does):

$$(i\partial\!\!\!/ - m)\psi \to (-i\partial\!\!\!/^* - m)\psi^* = 0$$

$$C(-i\partial\!\!\!/^* - m)\psi^* = (i\partial\!\!\!/ - m)C\psi^* = (i\partial\!\!\!/ - m)\psi^{(c)} = 0$$

With this definition of a charge conjugate spinor, the reality condition becomes: $\psi^{(c)} = \psi$.

In the Majorana basis $C = \mathbb{1}$, and in the chiral basis

$$C = i\gamma^2 = \begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}$$

$$\psi^{(c)} = i\gamma^2 \psi^* = \begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma_2 & 0 \end{pmatrix} \begin{pmatrix} u_+^* \\ u_-^* \end{pmatrix} = \begin{pmatrix} i\sigma^2 u_-^* \\ -i\sigma^2 u_+^* \end{pmatrix}$$

and the condition $\psi = \psi^{(c)}$ becomes $u_- = -i\sigma^2 u_+^*$, $u_+ = i\sigma^2 u_-^*$.

The latter is identical to the first since $u_+^* = -i\sigma^{2T} u_- \Rightarrow u_- = -i\sigma_2 u_+^*$ ✓

$$\sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \sigma_2 \sigma_2^T = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

So, we can replace u_- with $-i\sigma^2 u_+^*$ and write:

$$\psi = \begin{pmatrix} u_+ \\ -i\sigma^2 u_+^* \end{pmatrix}$$