Noether's theorem (reminder)

Infinitesimal transformations $\delta\phi_a(x) = X_a(\phi) \Rightarrow \delta\mathcal{L} = \partial_\mu F^\mu$

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \cdot X_a(\phi) - F^\mu(\phi)$$

Example: Energy-momentum tensor $T^\mu{}_\nu = (j^\mu)_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \cdot \partial_\nu\phi_a - \delta^\mu{}_\nu \mathcal{L}$

$\partial_\mu T^\mu{}_\nu = 0 \qquad \forall\, \nu = 0, \dots, 3$

$E = \int d^3x\, T^{00}$ and $P^i = \int d^3x\, T^{0i}$ are all conserved quantities

Symmetries and conserved currents

$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi$

$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_a)} = \bar{\psi} i\gamma^\mu ; \quad j^\mu = \bar{\psi} i\gamma^\mu \cdot \partial^\nu\psi - g^{\mu\nu}\mathcal{L}$

Spacetime translations:

$$\delta\psi = \varepsilon^\mu \partial_\mu \psi \Rightarrow T^{\mu\nu} = i\bar{\psi}\gamma^\mu\partial^\nu\psi - g^{\mu\nu}\mathcal{L}$$

Evaluated at the EoM we get: $\mathcal{L} = 0 \Rightarrow T^{\mu\nu} = i\bar{\psi}\gamma^\mu\partial^\nu\psi$

$$E = \int d^3x\, T^{00} = \int d^3x\, i\bar{\psi}\gamma^0\partial^0\psi = \int d^3x\, \bar{\psi}(-i\gamma^i\partial_i + m)\psi$$

Lorentz transf.:

$\psi^\alpha(x) \to S[\Lambda]^\alpha{}_\beta \psi^\beta(\Lambda^{-1}x) \qquad \Lambda^{-1}x = \left(\delta^\mu{}_\nu - \frac{1}{2}(\omega^\mu{}_\nu - \omega_\nu{}^\mu)\right)x^\nu$

$\Lambda^{\mu\nu} = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}M^{(\rho\sigma)}\right)^{\mu\nu} = \exp\left(\frac{1}{2}(\Omega^{\mu\nu} - \Omega^{\nu\mu})\right);$

$(M^{\rho\sigma})^\mu{}_\nu = g^{\rho\mu}\delta^\sigma{}_\nu - g^{\sigma\mu}\delta^\rho{}_\nu$

$$\delta\psi^\alpha = -\omega^\mu{}_\nu x^\nu \partial_\mu\psi^\alpha + \frac{1}{2}\Omega_{\rho\sigma}(S^{\rho\sigma})^\alpha{}_\beta\psi^\beta$$
$$= -\omega^{\mu\nu}\left[x_\nu\partial_\mu\psi^\alpha - \frac{1}{2}(S_{\mu\nu})^\alpha{}_\beta\psi^\beta\right]$$

Let us derive the form of the conserved current:

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_a)} = \bar{\psi} i\gamma^\mu \Rightarrow \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_a)} \cdot \delta\psi_a = \bar{\psi} i\gamma^\mu(-\omega^{\rho\sigma})\left[x_\sigma\partial_\rho\psi - \frac{1}{2}S_{\rho\sigma}\psi\right]$$
$$= \frac{\omega^{\rho\sigma}}{2}\Big[x_\rho\bar{\psi}i\gamma^\mu\partial_\sigma\psi - x_\sigma\bar{\psi}i\gamma^\mu\partial_\rho\psi$$
$$+ i\bar{\psi}\gamma^\mu S_{\rho\sigma}\psi\Big]$$

$$\Rightarrow J^\mu = x_\rho T^\mu{}_\sigma - x_\sigma T^\mu{}_\rho + i\bar{\psi}\gamma_\mu S_{\rho\sigma}\psi$$

Conserved current: $(J^\mu)^{\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho} \underset{\uparrow}{-} i\bar\psi\gamma^\mu S^{\rho\sigma}\psi$

From Tong, check the sign

Internal symmetries

Vector symm.: $\psi \to e^{-i\alpha}\psi \Rightarrow \quad j^\mu_V = \bar\psi\gamma^\mu\psi$

$$\delta\psi = -i\alpha\psi \qquad j^\mu_V = \bar\psi i\gamma^\mu(-i\psi) = \bar\psi\gamma^\mu\psi$$

$$\partial_\mu j^\mu_V = (\partial_\mu\bar\psi)\gamma^\mu\psi + \bar\psi\not\partial\psi = +im\bar\psi\psi - im\bar\psi\psi = 0$$

$$(i\not\partial - m)\psi = 0 \qquad \bar\psi(i\overleftarrow{\not\partial} + m) = 0$$

$$\Rightarrow Q = \int d^3x\, j^0_V = \int d^3x\, \psi^\dagger\psi$$

Axial symm. $\quad \psi \to e^{i\alpha\gamma^5}\psi \; ; \; \bar\psi \to \bar\psi e^{i\alpha\gamma_5} \Rightarrow j^\mu_A = \bar\psi\gamma^\mu\gamma_5\psi$

$$\partial_\mu j^\mu_A = (\partial_\mu\bar\psi)\gamma^\mu\gamma^5\psi + \bar\psi\gamma^\mu\gamma_5\partial_\mu\psi = -im\,\bar\psi\gamma_5\psi - \bar\psi\gamma_5\not\partial\psi =$$

$$= -im\,\bar\psi\gamma_5\psi - im\,\bar\psi\gamma_5\psi =$$

$$= 2im\bar\psi\gamma_5\psi \xrightarrow{m\to 0} 0$$

Plane wave solutions

$$(i\partial\!\!\!/ - m)\psi = 0$$

Ausatz: $\psi = u(\vec{p})\, e^{-ipx} \Rightarrow (p\!\!\!/ - m)u(\vec{p}) = \begin{pmatrix} -m & p^\mu \sigma_\mu \\ p^\mu \bar{\sigma}_\mu & -m \end{pmatrix} u(\vec{p}) = 0$

Claim: $u(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix}$ is a solution for any $\xi : \xi^\dagger \xi = 1$

Proof $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$:

$$p\cdot\sigma\, u_2 = m u_1$$

$$p\cdot\bar{\sigma}\, u_1 = m u_2$$

$$(p\cdot\sigma)(p\cdot\bar{\sigma}) = p_0^2 - p_i p_j \sigma^i \sigma^j = p_0^2 - \vec{p}^{\,2} = m^2$$

If I take $u_1 = (p\cdot\sigma)\xi'$ then

$$(p\cdot\bar{\sigma})(p\cdot\sigma)\xi' = m^2 \xi' = m u_2 \Rightarrow u_2 = m\xi'$$

$\Rightarrow u(\vec{p}) = A\begin{pmatrix} (p\cdot\sigma)\xi' \\ m\xi' \end{pmatrix}$ is a solution.

$\xi' = \sqrt{p\cdot\bar{\sigma}}\,\xi$ and $A = \frac{1}{m}$ gives us

$u(\vec{p}) = \frac{1}{m}\begin{pmatrix} (p\cdot\sigma)\sqrt{p\cdot\bar{\sigma}}\,\xi \\ m\sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix}$, but $\sqrt{(p\cdot\sigma)(p\cdot\bar{\sigma})} = m \Rightarrow u(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix}$

makes things symmetric and leads us to the desired solution.

Negative energy solutions:

$$\psi = v(\vec{p})e^{ipx}$$

$$(p\!\!\!/ + m)v(\vec{p}) = \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix}$$

$$v(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta \end{pmatrix}$$

is obtained from very similar manipulations.

Inner and outer products.

Let us introduce a basis for ξ and η: ξ_r, η_r , $r = 1,2$ and require that it is orthonormal: $\xi_r^\dagger \xi_s = \delta_{rs}$; $\eta_r^\dagger \eta_s = \delta_{rs}$,

for example $\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $\xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. These give two indep. solutions for the plane waves:

$$u_s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi_s \end{pmatrix} .$$

Inner products of these basis fermions:

$$u_r^\dagger(\vec{p})\cdot u_s(\vec{p}) = \left(\xi_r^\dagger \sqrt{p\cdot\sigma}, \xi_r^\dagger\sqrt{p\cdot\bar{\sigma}}\right) \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi_s \end{pmatrix} = \xi_r^\dagger (p\cdot\sigma)\xi_s + \xi_r^\dagger (p\cdot\bar{\sigma})\xi_s$$

$$= p^\mu \xi_r^\dagger (\sigma_\mu + \bar{\sigma}_\mu)\xi_s = p^\mu \xi_r^\dagger\, 2\delta_{\mu 0}\cdot \mathbb{1}\, \xi_s = 2p^0 \cdot \delta_{rs}$$

and for the Lorentz-invariant version:

$$\bar{u}_r(\vec{p})\cdot u_s(\vec{p}) = \left(\xi_r^\dagger \sqrt{p\cdot\sigma}, \xi_r^\dagger\sqrt{p\cdot\bar{\sigma}}\right) \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi_s \end{pmatrix} =$$

$$= \xi_r^\dagger \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}}\,\xi_s + \xi_r^\dagger \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\sigma}\,\xi_s =$$

$$= 2m\,\delta_{rs}$$

And similarly for the negative-energy solutions:

$$v_r^\dagger(\vec{p})\cdot v_s(\vec{p}) = 2p_0\,\delta_{rs}$$

$$\bar{v}_r(\vec{p})\cdot v_s(\vec{p}) = -2m\delta_{rs}$$

whereas the mixed products vanish:

$$\bar{u}_r(\vec{p})\, v_s(\vec{p}) = 0 = \bar{v}_r(\vec{p})\cdot u_s(\vec{p})$$

For the other inner product is more useful to take it for the opposite momentum:

$$u_r^\dagger(\vec{p})\cdot v_s(-\vec{p}) = v_r^\dagger(\vec{p})\cdot u_s(-\vec{p}) = 0$$

Finally for the outer products we have:

$$\sum_{s=1}^{2} u_s(\vec{p})\bar{u}_s(\vec{p}) = \not{p} + m$$

$$\sum_{s=1}^{2} v_s(\vec{p})\bar{v}_s(\vec{p}) = \not{p} - m$$

$$\sum_{s=1}^{2} \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi_s \end{pmatrix} \left(\xi_s^{+}\sqrt{p\cdot\bar{\sigma}} \,,\, \xi_s^{+}\sqrt{p\cdot\sigma} \right) =$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \; ; \; \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow \sum_{s=1}^{2} \xi_s = \mathbb{1}$ and overall we get:

$$\begin{pmatrix} m & p\cdot\sigma \\ p\bar{\sigma} & m \end{pmatrix} = \not{p} + m$$

and similarly for v_r

Quantization of the Dirac field.

In order to quantize the Dirac field we will first proceed naively, following the same steps as for the scalar field, and show that we will face inconsistencies.

We start from the conjugate momentum:

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\bar{\psi}\gamma^0 = i\psi^{+}$$

Remark: this is the reason why ψ^{+} does not count as indep. degrees of freedom: conjugate momenta are never counted as such.

$\Rightarrow$ canonical commutation relations:

$$[\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})] = [\psi_\alpha^{+}(\vec{x}), \psi_\beta^{+}(\vec{y})] = 0$$

$$[\psi_\alpha(\vec{x}), \psi_\beta^{+}(\vec{y})] = \delta_{\alpha\beta}\,\delta^3(\vec{x}-\vec{y})$$

$$\psi(\vec{x}) = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \left[b_{s\vec{p}}\, u_s(\vec{p}) e^{i\vec{p}\vec{x}} + c^{+}_{s,\vec{p}}\, v_s(\vec{p})\, e^{-i\vec{p}\vec{x}} \right]$$

$$\psi^{+}(\vec{x}) = \sum_{s=1} \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \left[c_{s,\vec{p}}\, v_s^{+}(\vec{p}) e^{i\vec{p}\cdot\vec{x}} + b^{+}_{s,\vec{p}}\, u_s^{+}(\vec{p})\, e^{-i\vec{p}\cdot\vec{x}} \right]$$

Let's see for what commutation relations for the c and b operators we can reproduce the canonical ones.

$$\left[\psi(\vec{x}), \psi^{+}(\vec{y})\right] = \sum_{r,s} \int \frac{d^3p\, d^3q}{(2\pi)^6\sqrt{4E_{\vec{p}}E_{\vec{q}}}} \Big\{ \left[b_{r,\vec{p}}, b^{+}_{s,\vec{q}}\right] u_r(\vec{p})\, u_s^{+}(\vec{q})\, e^{i\vec{p}\cdot\vec{x}-i\vec{q}\cdot\vec{y}} + \left[c^{+}_{r,\vec{p}}, c_{s,\vec{q}}\right] v_r(\vec{p})\, v_s^{+}(\vec{q})\, e^{-i\vec{p}\cdot\vec{x}+i\vec{q}\cdot\vec{y}} \Big\}$$

We can now try with $\left[b_{r,\vec{p}}, b^{+}_{s,\vec{q}}\right] = (2\pi)^3\, \delta_{rs}\, \delta^3(\vec{p}-\vec{q})$, however for $\left[c_{r,\vec{p}}, c^{+}_{s,\vec{q}}\right] = -(2\pi)^3\, \delta_{rs}\, \delta^3(\vec{p}-\vec{q})$ otherwise we will have a relative minus sign in the second term. In this way we get:

$$= \sum_{r} \int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}} \Big[\underbrace{u_r(\vec{p})\bar{u}_r(\vec{p})}_{\not{p}+m} \gamma_0\, e^{i\vec{p}(\vec{x}-\vec{y})} + \underbrace{v_r(\vec{p})\bar{v}_r(\vec{p})}_{\not{p}-m} \gamma_0\, e^{-i\vec{p}(\vec{x}-\vec{y})} \Big]$$

$$= \int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}} \Big[\underbrace{(\not{p}+m)\gamma_0\, e^{i\vec{p}(\vec{x}-\vec{y})}}_{(p_0\gamma^0 + p_i\gamma^i + m)\gamma_0 e^{i\vec{p}\cdot(\vec{x}-\vec{y})}} + \underbrace{(\not{p}-m)\gamma_0\, e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}}_{(p_0\gamma^0 - p_i\gamma^i - m)\gamma_0\, e^{i\vec{p}(\vec{x}-\vec{y})}} \Big]$$

← sign flip for $\vec{p}$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{\cancel{2\cdot p_0}}{\cancel{2E_{\vec{p}}}}\, e^{i\vec{p}(\vec{x}-\vec{y})} = \mathbb{1}\, \delta^3(\vec{x}-\vec{y})$$

as desired.

Hamiltonian

$$\mathcal{H} = \pi\cdot\dot{\psi} - \mathcal{L} = i\psi^{+}\cdot\dot{\psi} - \bar{\psi}(i\not{\partial}-m) = \bar{\psi}(\cancel{i\gamma^0\partial_0} - \cancel{i\gamma^0\partial_0} - i\gamma^i\partial_i + m)\psi$$

$$= \bar{\psi}(-i\gamma^i\partial_i + m)\psi$$

which agrees with T^{00} we calculated above with Noether.

We now want to express H as an operator, in terms of b and c:

$$\left(-i\gamma^i\partial_i+m\right)\psi=\sum_s\int\frac{d^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}}\left[b_{s,\vec{p}}\underbrace{\left(\gamma^i p_i+m\right)u_s(\vec{p})}_{\gamma^0 p_0\, u_s(\vec{p})}e^{i\vec{p}\cdot\vec{x}}+c^{+}_{s,\vec{p}}\underbrace{\left(-\gamma^i p_i+m\right)v_s(\vec{p})}_{-\gamma^0 p_0\, v_s(\vec{p})}e^{-i\vec{p}\cdot\vec{x}}\right]$$

$$=\sum_s\int\frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_{\vec{p}}}{2}}\left[b_{s,\vec{p}}\,\gamma^0 u_s(\vec{p})\,e^{i\vec{p}\cdot\vec{x}}-c^{+}_{s,\vec{p}}\,\gamma^0 v_s(\vec{p})\,e^{-i\vec{p}\cdot\vec{x}}\right]$$

Let us insert this expression in the integral for the Hamiltonian:

$$H=\int d^3x\,\bar{\psi}\left(-i\gamma^i\partial_i+m\right)\psi=\sum_{r,s}\int\frac{d^3x\,d^3p\,d^3q}{(2\pi)^6\sqrt{2E_{\vec{q}}}}\sqrt{\frac{E_{\vec{p}}}{2}}\left[b^{+}_{r,\vec{q}}\,u^{+}_r(\vec{q})e^{-i\vec{q}\vec{x}}+c_{r,\vec{q}}\,v^{+}_r(\vec{q})e^{i\vec{q}\vec{x}}\right]\times$$

$$\times\left[b_{s,\vec{p}}\,u_s(\vec{p})\,e^{i\vec{p}\vec{x}}-c^{+}_{s,\vec{p}}\,v_s(\vec{p})e^{-i\vec{p}\vec{x}}\right]$$

the integral over d^3x produces a δ-function in the momenta, which eliminates the integral over d^3q:

$$=\sum_{r,s}\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\Big[b^{+}_{r,\vec{p}}\,b_{s,\vec{p}}\,\underbrace{u^{+}_r(\vec{p})u_s(\vec{p})}_{2p_0\delta_{rs}}-b^{+}_{r,\vec{p}}\,c^{+}_{s,-\vec{p}}\,\underbrace{u^{+}_r(\vec{p})v_s(-\vec{p})}_{\overset{\shortparallel}{0}}$$

$$-c_{r,\vec{p}}\,c^{+}_{s,\vec{p}}\,\underbrace{v^{+}_r(\vec{p})v_s(\vec{p})}_{2p_0\delta_{rs}}+c_{r,\vec{p}}\,b_{s,-\vec{p}}\,\underbrace{v^{+}_r(\vec{p})u_s(-\vec{p})}_{\overset{\shortparallel}{0}}\Big]$$

$$=\sum_s\int\frac{d^3p}{(2\pi)^3}E_{\vec{p}}\left[b^{+}_{r,\vec{p}}\,b_{r,\vec{p}}-c^{+}_{r,\vec{p}}\,c_{r,\vec{p}}+(2\pi)^3\delta^3(\vec{0})\right]$$

With this expression for the Hamiltonian we obtain the following commutation relations:

$$\left[H,b^{+}_{s,\vec{q}}\right]=E_{\vec{q}}\,b^{+}_{s,\vec{q}}$$: a $b^{+}_{s,\vec{q}}|0\rangle$ is an eigenstate of the Hamiltonian with eigenvalue $E_{\vec{q}}$.

and also

$$\left[H,c^{+}_{s,\vec{q}}\right]=E_{\vec{q}}\,c^{+}_{s,\vec{q}}$$ " $c^{+}_{s,\vec{q}}|0\rangle$ " "

remember that $\left[c_{s,\vec{q}},c^{+}_{r,\vec{p}}\right]=-(2\pi)^3\delta_{rs}\,\delta^3(\vec{p}-\vec{q})$

However, the minus sign in the commutation relation also implies $\langle s,\vec{q}|r,\vec{p}\rangle=-\delta_{rs}\delta^3(\vec{p}-\vec{q})\Rightarrow$ in order to have states with a positive norm we should have interpreted $c_{r,\vec{q}}$ as a creation operator, which would have given us an expression for the Hamiltonian of the form ($d^+_{r,\vec q} = c_{r,\vec q}$):

$$\sum_s \int \frac{d^3p}{(2\pi)^3} E_{\vec p} \left[b^+_{r,\vec p}\, b_{r,\vec p} - d^+_{r,\vec p}\, d_{r,\vec p} \right]$$

so, no infinite energy for the vacuum, but a serious problem with negative-energy states:

$$\left[H, d^+_{s,\vec q}\right] = -E_{\vec q}\, d^+_{s,\vec q} \Rightarrow H d^+_{s,\vec q}|0\rangle = -E_{\vec q}\, d^+_{s,\vec q}|0\rangle$$

The only possible solution to this very serious problem is to instead adopt anticommutation relations for the fermion fields.

We will now proceed redoing the calculations with

$$\{\psi_\alpha(\vec x), \psi_\beta(\vec y)\} = \{\psi^+_\alpha(\vec x), \psi^+_\beta(\vec y)\} = 0$$

$$\{\psi_\alpha(\vec x), \psi^+_\beta(\vec y)\} = \delta_{\alpha\beta}\, \delta^3(\vec x - \vec y)$$

which are equivalent to:

$$\{b_{r,\vec p}, b^+_{s,\vec q}\} = \delta_{rs}\,(2\pi)^3\,\delta^3(\vec p - \vec q)$$

$$\{c_{r,\vec p}, c^+_{s,\vec q}\} = \delta_{rs}\,(2\pi)^3\,\delta^3(\vec p - \vec q)$$

$$\{b_{r,\vec p}, b_{s,\vec q}\} = \{b^+_{r,\vec p}, b^+_{s,\vec q}\} = \{c_{r,\vec p}, c_{s,\vec q}\} = \{c^+_{r,\vec p}, c^+_{s,\vec q}\} = 0$$

Repeating the same calculation for the Hamiltonian will lead us to the same expression, because so far we had not used any commutation relations.

$$H = \sum_r \int \frac{d^3p}{(2\pi)^3} E_{\vec p} \left[b^+_{r,\vec p}\, b_{r,\vec p} - c_{r,\vec p}\, c^+_{r,\vec p} \right]$$

minus sign! Opposite to that of bosons.

$$= \sum_r \int \frac{d^3p}{(2\pi)^3} E_{\vec p} \left[b^+_{r,\vec p}\, b_{r,\vec p} + c^+_{r,\vec p}\, c_{r,\vec p} - (2\pi)^3 \delta^3(\vec 0) \right]$$

Commutation relations of the $b^{(+)}$ and $c^{(+)}$ operators with the Hamiltonian:

$$\left[H, b^+_{s,\vec q}\right] = \sum_r \int \frac{d^3p}{(2\pi)^3} E_{\vec p} \left[b^+_{r,\vec p}\, b_{r,\vec p}, b^+_{s,\vec q} \right]$$

$[AB, C] = A\{B,C\} - \{A,C\}B$

$$\left[b^\dagger_{r,\vec{p}}\, b_{r,\vec{p}},\, b^\dagger_{s,\vec{q}}\right] = b^\dagger_{r,\vec{p}}\left\{b_{r,\vec{p}}, b^\dagger_{s,\vec{q}}\right\} - \cancelto{0}{\left\{b^\dagger_{r,\vec{p}}, b^\dagger_{s,\vec{q}}\right\}}\, b_{r,\vec{p}}$$

$$= b^\dagger_{r,\vec{p}}\;(2\pi)^3 \delta_{rs}\, \delta^3(\vec{p}-\vec{q})$$

$$\Rightarrow \quad \left[H, b^\dagger_{s,\vec{q}}\right] = E_{\vec{p}}\, b^\dagger_{s,\vec{q}}$$

Analogously one can show that $\left[H, b_{s,\vec{q}}\right] = -E_{\vec{p}}\, b_{s,\vec{q}}$, and the same for the $c^{(\dagger)}$ operators.

Note that two-particle states now have to satisfy:

$$b^\dagger_{r_1,\vec{p}_1}\, b^\dagger_{r_2,\vec{p}_2}\,|0\rangle = -\, b^\dagger_{r_2,\vec{p}_2}\, b^\dagger_{r_1,\vec{p}_1}\,|0\rangle$$

which agrees with Pauli's principle.

Propagators.

Heisenberg picture $\Rightarrow$ the fields become time dependent:

$$\frac{\partial \psi}{\partial t} = i\left[H, \psi\right]$$

$$\Rightarrow \quad \psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\vec{p}}}}\left[b_{s,\vec{p}}\, u_s(\vec{p})\, e^{-ipx} + c^\dagger_{s,\vec{p}}\, v_s(\vec{p})\, e^{ipx}\right]$$

and

$$\psi^\dagger(x) = \sum_s \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\vec{p}}}}\left[b^\dagger_{s,\vec{p}}\, u^\dagger_s(\vec{p})\, e^{ipx} + c_{s,\vec{p}}\, v^\dagger_s(\vec{p})\, e^{-ipx}\right]$$

Anticommutator of the fields:

$$i\, S_{\alpha\beta}(x-y) = \left\{\psi_\alpha(x), \bar{\psi}_\beta(y)\right\}$$

often written without indices:

$$iS(x-y) = \sum_{s,r} \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{\sqrt{4E_{\vec{p}}E_{\vec{q}}}}\left[\left\{b_{s,\vec{p}}, b^\dagger_{r,\vec{q}}\right\} u_s(\vec{p})\, \bar{u}_r(\vec{q})\, e^{-ipx+iqy} + \left\{c^\dagger_{s,\vec{p}}, c_{r,\vec{q}}\right\} v_s(\vec{p})\, \bar{v}_r(\vec{q})\, e^{ipx-iqy}\right]$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}}\left[(\not{p}+m)\, e^{-ip(x-y)} + (\not{p}-m)\, e^{ip(x-y)}\right]$$

$$= \left(i \not\partial_x + m\right)\left(D(x-y) - D(y-x)\right)$$

where $D(x-y) = \underset{\text{(Coleman)}}{\Delta_+(x-y)} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)}$

What do we learn from this?

- From the bosonic theory we have learned that

$$D(x-y) - D(y-x) = 0 \qquad \forall\ (x-y)^2 < 0$$

$$\Rightarrow [\phi(x), \phi(y)] = 0$$ which we interpreted as proof that our theory is causal.

Now we have: $$\{\psi_a(x), \bar{\psi}_b(y)\} = 0 \qquad \forall\ (x-y)^2 < 0$$

whereas $[\psi_a(\vec{x},t), \psi_b(\vec{y},t)] = 2\psi_a(\vec{x},t)\psi_b(\vec{y},t) \neq 0$ ⇒ they cannot be observables.

Fermion bilinears like $\bar{\psi}\Gamma\psi$ are instead observables and do commute at spacelike intervals as we will see later.

- We also note that

$$\left(i\not\partial_x - m\right) S(x-y) = -\left(\Box + m^2\right)\left[D(x-y) - D(y-x)\right] = 0$$

so $S(x-y)$ satisfies Dirac's equation.