# Feynman's propagator for a fermion field.

$$\psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \Big[ \underbrace{b_{s\vec{p}}\, u_s(\vec{p})\, e^{-ipx}}_{\psi^{(+)}} + \underbrace{c^\dagger_{s\vec{p}}\, v_s(\vec{p})\, e^{ipx}}_{\psi^{(-)}} \Big]$$

$$\bar{\psi}(x) = \sum_s \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \Big[ \underbrace{b^\dagger_{s\vec{p}}\, \bar{u}_s(\vec{p})\, e^{ipx}}_{\bar{\psi}^{(-)}} + \underbrace{c_{s\vec{p}}\, \bar{v}_s(\vec{p})\, e^{-ipx}}_{\bar{\psi}^{(+)}} \Big]$$

$$\begin{aligned} iS(x-y) &= \{\psi(x), \bar{\psi}(y)\} \\ &= \{\psi^{(+)}, \bar{\psi}^{(-)}\} + \{\psi^{(-)}, \bar{\psi}^{(+)}\} \\ &= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \Big[ (\not{p}+m)\, e^{-ip(x-y)} + (\not{p}-m)\, e^{ip(x-y)} \Big] \\ &= (i\not{\partial}+m)\big[ D(x-y) - D(y-x) \big] \end{aligned}$$

## Time-ordered product:

The definition of the T-operator must be adapted to fermions:

$$T(\psi_a(x)\psi_b(y)) = \theta(x_0-y_0)\psi_a(x)\psi_b(y) - \theta(y_0-x_0)\psi_b(y)\psi_a(x)$$

Plot like Fig. 21.1 in Coleman.

For many fields

$$T(\psi_1(x_1)\cdots\psi_n(x_n)) = (-1)^P \psi_{j_1}(x_{j_1})\cdots\psi_{j_n}(x_{j_n}) \qquad \text{for } x^0_{j_1} > x^0_{j_2} \cdots x^0_{j_n}$$

We can now define the Feynman's propagator:

$$S_F(x-y) = \langle 0|T\psi(x)\bar{\psi}(y)|0\rangle \equiv \begin{cases} \langle 0|\psi(x)\bar{\psi}(y)|0\rangle & x^0 > y^0 \\ -\langle 0|\bar{\psi}(y)\psi(x)|0\rangle & y^0 > x^0 \end{cases}$$

For $x^0 > y^0$ we have

$$\langle 0|T\psi(x)\bar{\psi}(y)|0\rangle = \int \frac{d^3p\, d^3q}{(2\pi)^6\sqrt{4E_{\vec{p}}E_{\vec{q}}}}\, e^{-ipx+iqy} \sum_{r,s} \underbrace{\underbrace{\langle 0|b_{r,\vec{p}}\, b^{+}_{s,\vec{q}}|0\rangle}_{\langle 0|\{b_{r,\vec{p}},\, b^{+}_{s,\vec{q}}\}|0\rangle}}_{(2\pi)^3\delta^3(\vec{p}-\vec{q})\,\delta_{rs}}\, u_r(\vec{p})\bar{u}_s(\vec{q})$$

$$\text{"}\not{p}+m\text{"}$$

$$= \int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}}\, e^{-ip(x-y)}(\not{p}+m) = (i\not{\partial}+m)\int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}}\, e^{-ip(x-y)}$$

For $y^0 > x^0$ we get

$$-\int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}}\, e^{ip(x-y)}(\not{p}-m) = (i\not{\partial}_x+m)\int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}}\, e^{ip(x-y)} \quad p \to -p$$

$$= (i\not{\partial}_x+m)\int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}}\, e^{-ip(x-y)}$$

So that:

$$S_F(x-y) = (i\not{\partial}+m)\Delta_F(x-y) = (i\not{\partial}+m)\, i\int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x-y)}}{p^2-m^2+i\varepsilon} = i\int \frac{d^4p}{(2\pi)^4}\frac{\not{p}+m}{p^2-m^2+i\varepsilon}\, e^{-ip(x-y)}$$

Notice that $(i\not{\partial}_x - m)S_F(x-y) = (i\not{\partial}_x - m)(i\not{\partial}_x+m)\Delta_F(x-y) =$

$$= -(\Box+m^2)\Delta_F(x-y) = i\delta^4(x-y)$$

which means that $S_F(x-y)$ is the Green's function of Dirac's operator, just like $\Delta_F(x-y)$ is the Green's function of Klein-Gordon's operator.

So, if we introduce the contraction for fermions analogously to that for scalars:

$$\overbrace{\psi_1(x)\bar{\psi}_2(y)} = \theta(x_0-y_0)\{\psi_1^{(+)}(x), \bar{\psi}_2^{(-)}(y)\} - \theta(y_0-x_0)\{\bar{\psi}_2^{(+)}(y), \psi_1^{(-)}(x)\}$$

then we have:

$$\overbrace{\psi(x)\bar{\psi}(y)} = (i\not{\partial}+m)\,\overbrace{\varphi(x)\varphi(y)}$$

## Wick's theorem.

Normal order for fermions:

$$:\psi_1(x_1)\,\psi_2(x_2): \;=\; \psi_1^{(+)}(x_1)\,\psi_2^{(+)}(x_2) + \psi_1^{(-)}(x_1)\,\psi_2^{(+)}(x_2)$$
$$+\, \psi_1^{(-)}(x_1)\,\psi_2^{(-)}(x_2) - \psi_2^{(-)}(x_2)\,\psi_1^{(+)}(x_1)$$

Let us now look at the difference between time-ordered product and normal-ordered one and show that

$$\overbrace{\psi_1(x)\,\bar\psi_2(y)} = T(\psi_1(x)\,\bar\psi_2(y)) - :\psi_1(x)\,\bar\psi_2(y):$$

For $x^0 > y^0$

$$T(\psi_1(x)\bar\psi_2(y)) = \psi_1^{(+)}(x)\,\bar\psi_2^{(+)}(y) + \psi_1^{(-)}(x)\,\bar\psi_2^{(+)}(x_2)$$
$$+\,\psi_1^{(-)}(x)\,\bar\psi_2^{(-)}(y) + \psi_1^{(+)}(x)\,\bar\psi_2^{(-)}(y)$$

so that

$$\overbrace{\psi_1(x)\,\bar\psi_2(y)} = \left\{\psi_1^{(+)}(x), \bar\psi_2^{(-)}(y)\right\} \qquad x^0 > y^0 \qquad \checkmark$$

For $y^0 > x^0$ we have

$$\overbrace{\psi_1(x)\,\bar\psi_2(y)} = -\left\{\bar\psi_2^{(+)}(y), \psi_1^{(-)}(x)\right\}$$

and also:

$$T(\psi_1(x)\bar\psi_2(y)) \overset{(y^0>x^0)}{=} -\bar\psi_2^{(+)}(y)\,\psi_1^{(+)}(x) - \bar\psi_2^{(-)}(y)\,\psi_1^{(+)}(x) - \bar\psi_2^{(-)}(y)\,\psi_1^{(-)}(x) - \bar\psi_2^{(+)}(y)\,\psi_1^{(-)}(x)$$

whereas $:\psi_1(x)\bar\psi_2(y):$ does not need to change. But since $:\bar\psi_2(y)\,\psi_1(x): = -:\psi_1(x)\,\bar\psi_2(y):$ we have

$$-:\psi_1(x)\,\bar\psi_2(y): \;=\; :\bar\psi_2(y)\,\psi_1(x):$$

$$= \bar\psi_2^{(+)}(y)\,\psi_1^{(+)}(x) + \bar\psi_2^{(-)}(y)\,\psi_1^{(+)}(x) + \bar\psi_2^{(-)}(y)\,\psi_1^{(-)}(x) - \psi_1^{(-)}(x)\,\bar\psi_2^{(+)}(y)$$

and all in all we get

$$\contraction{}{\psi}{_1(x)}{\bar{\psi}}\psi_1(x)\bar{\psi}_2(y) = T(\psi_1(x)\bar{\psi}_2(y)) + :\bar{\psi}_2(y)\psi_1(x):$$
$$- \left\{\bar{\psi}_2^{(+)}(y)\, \psi_1^{(-)}(x)\right\} \qquad \text{q.e.d.}$$

Taking the matrix element between two vacuum states we obtain again that the contraction is nothing but Feynman's propagator:

$$\langle 0|\, \contraction{}{\psi}{_1(x)}{\bar{\psi}}\psi_1(x)\bar{\psi}_2(y)|0\rangle = \langle 0|\, T\,\psi_1(x)\bar{\psi}_2(y)|0\rangle$$

The statement of Wick's theorem is exactly the same as in the case of scalars, with the caveat that like for complex scalar fields the contraction can only happen between $\psi$ and $\bar{\psi}$:

$$\contraction{}{\psi}{(x)}{\psi}\psi(x)\psi(y) = \contraction{}{\bar{\psi}}{(x)}{\bar{\psi}}\bar{\psi}(x)\bar{\psi}(y) = 0 .$$

So, if we want to evaluate the S-matrix element for a certain process in perturbation theory, we need to consider the appropriate term in the expansion of $T\exp\left\{i\int d^4x\, \mathcal{H}_I(x)\right\}$;

say
$$\int d^4x_1 \ldots d^4x_n\, T\left[\mathcal{H}_I(x_1) \ldots \mathcal{H}_I(x_n)\right]$$

and calculate the matrix element between the states

$$\langle \vec{p}\,'_1, \ldots \vec{p}\,'_m; (s'_1, \vec{q}\,'_1), \ldots (s'_M\, \vec{q}\,'_M)| \quad \text{and} \quad |\vec{p}_1, \ldots \vec{p}_n; (s_1, \vec{q}_1) \ldots; (s_N, \vec{q}_N)\rangle$$

To do this we have to calculate all possible contractions among the operators between the vacuum states to the left and to the right, so taking into account also the creation and annihilation operators coming from the external states.

What remains the same is that all integration over $x$ coming from the expansion of $T\left\{\exp i\int d^4x\, \mathcal{H}_I(x)\right\}$ generate $\delta$-functions for momentum conservation at each vertex and an overall one.
The simplest way to carry out these calculations is with the help of Feynman rules and diagrams, which have to be adapted to the case of fermions. But before deriving them, let's see how they emerge with the help of one concrete example:

## Yukawa theory:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}M^2\varphi^2 + \bar{\psi}(i\partial\!\!\!/ - m)\psi - \lambda\varphi\bar{\psi}\psi$$

What is the dimension of the coupling $\lambda$?

$$[\bar{\psi}(i\partial\!\!\!/ - m)\psi] = 4 \quad \Rightarrow \quad [\psi] = 3/2$$

$$[\varphi\bar{\psi}\psi] = 1 + 3/2 + 3/2 = 4 \quad \Rightarrow \quad [\lambda] = 0$$

Let us calculate the amplitude of the $NN \to NN$ scattering process:

$$|i\rangle = \sqrt{4E_{\vec{p}}\,E_{\vec{q}}}\; b^+_{s,\vec{p}}\, b^+_{r,\vec{q}}\,|0\rangle$$

$$|f\rangle = \sqrt{4E_{\vec{p}'}\,E_{\vec{q}'}}\; b^+_{s',\vec{p}'}\, b^+_{r,\vec{q}'}\,|0\rangle$$

$$\langle f| = \langle 0|\, b_{r,\vec{p}'}\, b_{s,\vec{q}'} \cdot \sqrt{4E_{\vec{p}'}\,E_{\vec{q}'}}$$

In order to have a nonzero contribution we need to reach second order in the expansion of

$$T\exp\left\{-i\lambda\int d^4x\, \varphi(x)\bar{\psi}(x)\psi(x)\right\}$$

because we need two $\psi$ and two $\bar{\psi}$ fields.

$$\frac{(-i\lambda)^2}{2}\int d^4x\, d^4y\; \underbrace{T\left[\varphi(x)\bar{\psi}(x)\psi(x)\,\varphi(y)\bar{\psi}(y)\psi(y)\right]}_{\varphi(x)\varphi(y)\; :\bar{\psi}(x)\psi(x)\bar{\psi}(y)\psi(y):}$$

The action of the normal-ordered product of the fermion fields on the external states can be discussed as follows: it is the $\psi$ field which give a nonzero contribution when acting on $|i\rangle$:

$$-\int\frac{d^3k_1\, d^3k_2}{(2\pi)^6\sqrt{4E_{\vec{k}_1}E_{\vec{k}_2}}}\left[\bar{\psi}(x)\,u_m(\vec{k}_1)\right]\left[\bar{\psi}(y)u_n(\vec{k}_2)\right]e^{-ik_1x-ik_2y}\; b_{m,\vec{k}_1}b_{n,\vec{k}_2}b^+_{s,\vec{p}}b^+_{r,\vec{q}}|0\rangle$$

anticomm. of $b_{m,\vec{k}_1}$ with $\bar{\psi}(y)$

we then have

$$b_{m,\vec{k}_1}b_{n,\vec{k}_2}b^+_{s,\vec{p}}b^+_{r,\vec{q}}|0\rangle = \left[b_{m,\vec{k}_1}\{b_{n,\vec{k}_2},b^+_{s,\vec{p}}\}b^+_{r,\vec{q}} - b_{m,\vec{k}_1}b^+_{s,\vec{p}}b_{n,\vec{k}_2}b^+_{r,\vec{q}}\right]|0\rangle$$

$$= \delta_{ns}(2\pi)^3\delta^3(\vec{k}_2-\vec{p})\{b_{m,\vec{k}_1},b^+_{r,\vec{q}}\}|0\rangle - \{b_{m,\vec{k}_1},b^+_{s,\vec{p}}\}\{b_{n,\vec{k}_2},b^+_{r,\vec{q}}\}|0\rangle =$$

$$=\left[\delta_{ns}\delta_{mr}(2\pi)^6\delta^3(\vec{k}_2-\vec{p})\delta^3(\vec{k}_1-\vec{q}) - \delta_{ms}\delta_{nr}(2\pi)^6\delta^3(\vec{k}_1-\vec{p})\delta^3(\vec{k}_2-\vec{q})\right]|0\rangle$$

$$\Rightarrow -\frac{1}{\sqrt{4E_{\vec{p}}E_{\vec{q}}}}\left\{\left[\bar{\psi}(x)u_r(\vec{q})\right]\left[\bar{\psi}(y)u_s(\vec{p})\right]e^{-iqx-ipy} - \left[\bar{\psi}(x)u_s(\vec{p})\right]\left[\bar{\psi}(y)u_r(\vec{q})\right]e^{-ipx-iqy}\right\}|0\rangle$$

We now multiply this from the left with $\langle f|$ and obtain:

$$\langle 0|b_{r,\vec{q}'}b_{s,\vec{p}'}\left[\bar{\psi}(x)\cdot u_r(\vec{q})\right]\left[\bar{\psi}(y)\cdot u_s(\vec{p})\right]|0\rangle \quad - \quad \text{the same with } (r,\vec{q})\leftrightarrow(s,\vec{p})$$

Now we should do the same manipulations done with the initial state, leading to:

$$\frac{2}{\sqrt{4E_{\vec p'}E_{\vec q'}}}\left[\bar u_{r'}(\vec q')\cdot u_r(\vec q)\;\bar u_{s'}(\vec p')\cdot u_s(\vec p)\,e^{ix(q'-q)+iy(p'-p)} - \bar u_{s'}(\vec p')u_r(\vec q)\;\bar u_{r'}(\vec p')u_s(\vec p)\,e^{ix(p'-q)+iy(q'-p)}\right]$$

The factor 2 cancels with the one coming from the expansion of the exponential, and the square roots in the denominator cancels with the ones from the state normalization. Finally we have to include the meson's propagator:

$$(-i\lambda)^2\int\frac{d^4x\,d^4y\,d^4k}{(2\pi)^4}\frac{i e^{ik(x-y)}}{k^2-M^2+i\varepsilon}\left[\bar u_{r'}(\vec q')\cdot u_r(\vec q)\;\bar u_{s'}(\vec p')\cdot u_s(\vec p)\,e^{ix(q'-q)+iy(p'-p)} - \bar u_{s'}(\vec p')u_r(\vec q)\;\bar u_{r'}(\vec p')u_s(\vec p)\,e^{ix(p'-q)+iy(q'-p)}\right]$$

From the integrals over $x$ and $y$ we get:

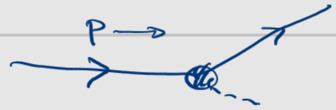
$$(-i\lambda)^2\int d^4k\,\frac{(2\pi)^4}{k^2-M^2+i\varepsilon}\left[\bar u_{r'}(\vec q')\cdot u_r(\vec q)\;\bar u_{s'}(\vec p')\cdot u_s(\vec p)\,\delta^4(k+q'-q)\,\delta^4(k-p'+p) - \bar u_{s'}(\vec p')u_r(\vec q)\;\bar u_{r'}(\vec p')u_s(\vec p)\,\delta^4(k+p'-q)\,\delta^4(k-q'+p)\right]$$

$$=(-i\lambda)^2(2\pi)^4\delta^4(p+q-p'-q')\left[\frac{[\bar u_{r'}(\vec q')u_r(\vec q)][\bar u_{s'}(\vec p')\cdot u_s(\vec p)]}{(p-p')^2-M^2+i\varepsilon}-\frac{[\bar u_{s'}(\vec p')\cdot u_r(\vec q)][\bar u_{r'}(\vec p')\cdot u_s(\vec p)]}{(p-q')^2-M^2+i\varepsilon}\right]$$

## Feynman rules for fermions:

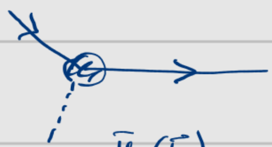
- Ext. fermions:

incoming fermion




$u_r(\vec p)$

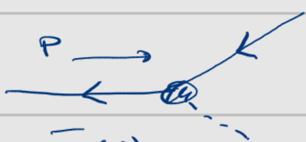
outgoing fermion



$\bar u_r(\vec p)$

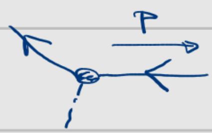
- Ext. antifermions:

incoming anti-fermion



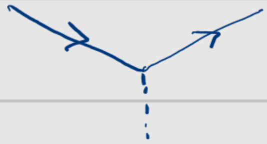

$\bar v_r(\vec p)$

outgoing anti-fermion



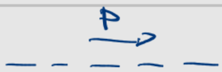

$v_r(\vec p)$

- Vertex



$$-i\lambda$$

- Internal lines:



$$\frac{i}{p^2-M^2+i\varepsilon}$$

$$\frac{i(\not p+m)}{p^2-m^2+i\varepsilon} = \frac{i}{\not p-m+i\varepsilon}$$

$$p^2-m^2+i\varepsilon = \left(\not p-(m+i\varepsilon)\right)\left(\not p+(m+i\varepsilon)\right)$$

- Momentum conservation must be imposed at each vertex.

- Signs due to statistics need to be taken care of.