Lagrangian for classical electrodynamics:

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

with $\quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

EoM: $\quad \partial_\mu \left( \frac{\partial L}{\partial \partial_\mu A_\nu} \right) = -\partial_\mu F^{\mu\nu}$

$$L = -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = -\frac{1}{2} F^{\mu\nu} \partial_\mu A_\nu$$

$$= -\frac{1}{2}\left(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu\right)$$

$$\Rightarrow \quad \frac{\partial L}{\partial \partial_\mu A_\nu} = -\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)$$

$$\partial_\mu \frac{\partial L}{\partial \partial_\mu A_\nu} = -\partial_\mu F^{\mu\nu}$$

From the definition of $F_{\mu\nu}$, it also satisfies the Bianchi identity:

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$$

$$\cancel{\partial_\lambda \partial_\mu A_\nu} - \cancel{\partial_\lambda \partial_\nu A_\mu} + \cancel{\partial_\mu \partial_\nu A_\lambda} - \cancel{\partial_\mu \partial_\lambda A_\nu} + \cancel{\partial_\nu \partial_\lambda A_\mu} - \cancel{\partial_\nu \partial_\mu A_\lambda} = 0$$

which follows just from the definition of $F_{\mu\nu}$.

Electromagnetic fields: $\quad A^\mu = (\phi, \vec{A})$

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \quad ; \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

$$E_i = -\partial_i A_0 + \partial_0 A_i = -F_{i0} = F_{0i}$$

and

$$\vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow B_i = \varepsilon_{ijk} \partial_j A^k \quad ; \quad B_1 = \partial_2 A^3 - \partial_3 A^2 = -\left(\partial_2 A_3 - \partial_3 A_2\right)$$

$$= -F_{23}$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$F_{ij} = \varepsilon_{ijk} B_k = \varepsilon_{ijk} \varepsilon_{klm} \partial_l A^m$$

$$= \partial_i A^j - \partial_j A^i =$$

Bianchi identity:

$$\partial_3 F_{12} + \partial_1 F_{23} + \partial_2 F_{31} = -\partial_z B_z - \partial_x B_x - \partial_y B_y = -\vec{\nabla}\cdot\vec{B} = 0$$

$$\partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} = \partial_0(-B_z) - \partial_x E_y + \partial_y E_x$$

$$= -\left[\partial_t B_z + \left(\partial_x E_y - \partial_y E_x\right)\right] = -\left[\partial_t B_z + (\vec{\nabla}\times\vec{E})_z\right]$$

with the other combinations we get the remaining two components.

All in all, the B.I. give $\boxed{\vec{\nabla}\cdot\vec{B} = 0 \text{ and } \frac{\partial\vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E}}$

The EoM, expressed in $\vec{E}$ and $\vec{B}$ are:

$$\partial_0 F^{0i} + \partial_1 F^{1i} + \partial_2 F^{2i} + \partial_3 F^{3i} = 0$$

$$\partial_t \vec{E}_i + \underbrace{\left(\partial_1 \varepsilon^{1ik} + \partial_2 \varepsilon^{2ik} + \partial_3 \varepsilon^{3ik}\right)}_{\varepsilon^{ikj}\partial_j B_k} B_k = 0$$

$$\partial_t \vec{E}_i - (\vec{\nabla}\times\vec{B})_i = 0 \Rightarrow \boxed{\partial_t \vec{E} = \vec{\nabla}\times\vec{B}}$$

and $$\partial_\mu F^{\mu 0} = -\partial_i F^{i0} = \partial_i F^{0i} = \boxed{\vec{\nabla}\cdot\vec{E} = 0}$$

EoM + B.I. $\Rightarrow$ Maxwell equations.

How many dofs are there in M.E. or classical electrodynamics?

Remarks:

1. There is no time derivative of $A_0$ in the Lagrangian $\Rightarrow$ no conjugate momentum $\Rightarrow$ no time evolution of $A_0$:

$$\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \partial_i F_{0i} = \partial_i(\partial_0 A_i - \partial_i A_0) = -\Delta A_0 - \partial_t \vec{\nabla}\cdot\vec{A} = 0$$

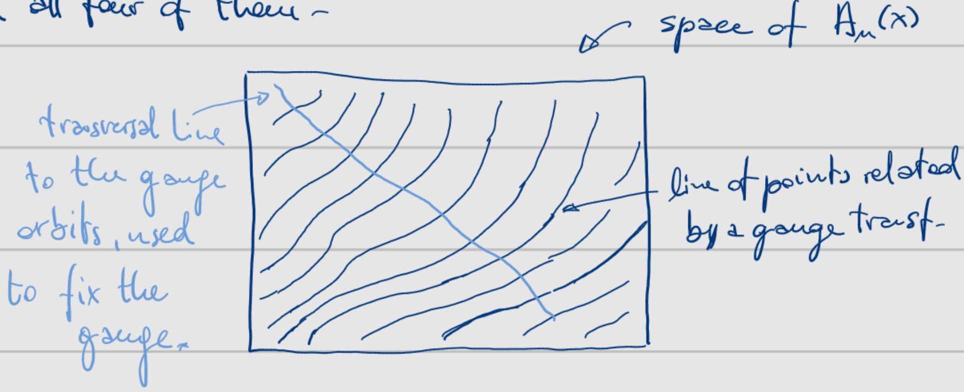
Solution: $$A_0(\vec{x},t) = \frac{1}{4\pi}\int d^3y \, \frac{\partial_t \vec{\nabla}\cdot\vec{A}(\vec{y},t)}{|\vec{x}-\vec{y}|}$$

At the initial time $t_0$ we need to know $\vec{A}(\vec{x},t_0)$, $\dot{\vec{A}}(\vec{x},t_0)$, $\forall \vec{x}$, but once we know these, $A_0(\vec{x},t_0)$ is also known. The solution $\vec{A}(\vec{x},t)$ for $t>t_0$ also determines $A_0(\vec{x},t)$.

2. The Lagrangian above is only written in terms of $F_{\mu\nu}$ which is invariant under:

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu \lambda(x) \qquad \text{for any } \lambda(x)$$

Two configurations $A_\mu(x)$ related by a gauge transformation are physically equivalent. The description in terms of $A_\mu(x)$, i.e. four functions, one of which is dependent, is clearly redundant: only two of the four are independent. On the other hand there is no easy way to get rid of simply one of the four because a gauge transformation involves in general all four of them.




Gauges:

Lorenz gauge: $\partial_\mu A^\mu = 0$

if $\partial_\mu A^\mu \neq 0$, let's call its nonzero value $f(x)$: $\partial_\mu A^\mu(x) = f(x)$

$$A'^\mu(x) = A^\mu(x) + \partial^\mu \lambda(x) \qquad \partial_\mu A'^\mu(x) = f(x) + \Box\lambda(x)$$

$$\Box\lambda(x) = -f(x) \quad \text{which always has a solution.}$$

But since $\Box\lambda'(x) = 0$ also has always a solution, the Lorenz gauge doesn't identify one specific gauge, but rather a class of gauges.

Coulomb gauge. $\vec{\nabla}\cdot\vec{A}=0 \implies A_0=0$.

Lorentz invariance is broken.

## Quantization of the em. field

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0$$

$$\pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = -F^{0i} \equiv E^i \qquad \left( E_i = -\partial_i A_0 + \dot{A}_i \right)$$

Hamiltonian: $\quad$ (remember: $\dot{A}_i = +\vec{\nabla}\cdot A_0 + E_i$)

$$H = \int d^3x \left[ \pi^i \dot{A}_i - \mathcal{L} \right] = \int d^3x \left[ \vec{E}\cdot\vec{E} - \tfrac{1}{2}\vec{E}\cdot\vec{E} + \tfrac{1}{2}\vec{B}\cdot\vec{B} + \vec{\nabla}\cdot A_0\cdot\vec{E} \right]$$

$$= \int d^3x \left[ \frac{1}{2}\left( \vec{E}\cdot\vec{E} + \vec{B}\vec{B} \right) - \underbrace{A_0}_{\text{Lagrange multiplier imposing } \vec{\nabla}\cdot\vec{E}=0} \vec{\nabla}\vec{E} \right]$$

Upon using:

$$-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} = -\frac{1}{4}\left( F^{0i}F_{0i} + F^{i0}F_{i0} + F^{ij}F_{ij} \right)$$

$$= \frac{1}{2}\left( \vec{E}\cdot\vec{E} - \vec{B}\cdot\vec{B} \right)$$

## Quantization in the Coulomb gauge.

$$\partial_\mu F^{\mu\nu} = 0 \implies \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu = 0$$

$$\Box A^\nu - \partial^\nu \big( \underbrace{\dot{A}^0}_{=0} - \underbrace{\vec{\nabla}\cdot\vec{A}}_{=0} \big) = 0$$

$$\implies \Box \vec{A} = 0$$

Solution:

$$\vec{A}(x) = \int \frac{d^3p}{(2\pi)^3} \vec{\xi}(\vec{p})\, e^{-ipx} \quad , \quad p_0^2 = |\vec{p}|^2$$

moreover $\vec{\nabla}\cdot\vec{A}=0 \Rightarrow \vec{\xi}\cdot\vec{p}=0$

Choose a basis in the plane orthogonal to $\vec{p}$: $\vec{\varepsilon}_r(\vec{p})\cdot\vec{p}=0$, $r=1,2$

$$\vec{\varepsilon}_r(\vec{p})\cdot\vec{\varepsilon}_s(\vec{p})=\delta_{rs}$$

Poisson brackets $\to$ commutators:

$$[A_i(\vec{x}),A_j(\vec{y})]=[E^i(\vec{x}),E^j(\vec{y})]=0$$

$$[A_i(\vec{x}),E^j(\vec{y})]=i\,\delta_i^j\,\delta^3(\vec{x}-\vec{y})$$

here we face a problem since $\vec{\nabla}\cdot\vec{A}=\vec{\nabla}\cdot\vec{E}=0$, but

$$[\vec{\nabla}\cdot\vec{A}(\vec{x}),\vec{\nabla}\cdot\vec{E}(\vec{y})]=-i\Delta_x\,\delta^3(\vec{x}-\vec{y})\neq 0$$

What's the problem? Quantization of a constrained system. Solution:

$$[A_i(\vec{x}),E_j(\vec{y})]=i\left(\delta_{ij}-\frac{\partial_i\partial_j}{\Delta}\right)\delta^3(\vec{x}-\vec{y})$$

$$=i\int\frac{d^3p}{(2\pi)^3}\left(\delta_{ij}-\frac{p_ip_j}{|\vec{p}|^2}\right)e^{i\vec{p}(\vec{x}-\vec{y})}$$

$$\Rightarrow [\vec{\nabla}\vec{A}(\vec{x}),E_j(\vec{y})]=i\int\frac{d^3p}{(2\pi)^3}\left(ip_j-ip_j\right)e^{i\vec{p}(\vec{x}-\vec{y})}=0$$

$$\vec{A}(\vec{x})=\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2|\vec{p}|}}\sum_{r=1}^{2}\vec{\varepsilon}_r(\vec{p})\left[a_{r,\vec{p}}\,e^{i\vec{p}\cdot\vec{x}}+a^\dagger_{r,\vec{p}}\,e^{-i\vec{p}\vec{x}}\right]$$

$$\vec{E}(\vec{x})=\int\frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{|\vec{p}|}{2}}\sum_{r=1}^{2}\vec{\varepsilon}_r(\vec{p})\left[a_{r,\vec{p}}\,e^{i\vec{p}\cdot\vec{x}}-a^\dagger_{r,\vec{p}}\,e^{-i\vec{p}\cdot\vec{x}}\right]$$

with $\vec{\varepsilon}_r(\vec{p})\cdot\vec{p}=0$ ; $\vec{\varepsilon}_r(\vec{p})\cdot\vec{\varepsilon}_s(\vec{p})=\delta_{rs}$

and from the commutation relations one can derive

$$[a_{r,\vec{p}},a_{s,\vec{q}}]=[a^\dagger_{r,\vec{p}},a^\dagger_{s,\vec{q}}]=0$$

$$[a_{r,\vec{p}},a^\dagger_{s,\vec{q}}]=(2\pi)^3\,\delta_{rs}\,\delta^3(\vec{p}-\vec{q})$$

which relies on the completeness relation

$$\sum_{r=1}^{2} \varepsilon_r(\vec{p})_i\, \varepsilon_r(\vec{p})_j = \delta_{ij} - \frac{p_i p_j}{|\vec{p}|^2}$$

Multiply from left and from right with any of the vector $\vec{\varepsilon}_r(\vec{p})$, $\vec{p}$ to confirm the relation.

Finally one can insert the expression for $\vec{E}$ and $\vec{A}$ in terms of $a_r(\vec{p})$ and $a_r^+(\vec{p})$ in the expr. for the Hamiltonian to obtain:

$$H = \int \frac{d^3p}{(2\pi)^3}\, |\vec{p}| \sum_{r=1}^{2} a^+_{r,\vec{p}}\, a_{r,\vec{p}}$$

## Lorentz gauge

$$\partial_\mu A^\mu = 0 \quad \Rightarrow \partial_\mu F^{\mu\nu} = \Box A^\nu - \partial^\nu \underbrace{\partial_\mu A^\mu}_{\overset{!}{0}} = 0$$

EoM: $\quad \Box A^\nu = 0$

This can be achieved by starting from a different Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_\mu A^\mu)^2$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu} = 0 \quad ; \quad \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu} = -F^{\mu\nu} - \frac{1}{2} \frac{\partial}{\partial \partial_\mu A_\nu} \left(\partial_\mu A_\nu g^{\mu\nu}\right)^2$$

$$= -F^{\mu\nu} - g^{\mu\nu} \partial_\rho A^\rho$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu} = -\partial_\mu F^{\mu\nu} - \partial^\nu \partial_\mu A^\mu$$

$$= -\Box A^\nu + \cancel{\partial^\nu \partial_\mu A^\mu} - \cancel{\partial^\nu \partial_\mu A^\mu} = 0$$

Conjugate momenta:

$$\pi^0 = \frac{\partial L}{\partial \dot{A}_0} = \frac{\partial}{\partial \dot{A}_0}\left(-\frac{1}{2}(\partial_\mu A^\mu)^2\right) = -\partial_\mu A^\mu$$

$$\pi^i = \frac{\partial L}{\partial \dot{A}_i} = \frac{\partial}{\partial \partial_0 A_i}\left(-\frac{1}{2}\partial_0 A_i F^{0i} + \dots\right) = \partial^i A^0 - \partial^0 A^i$$

Canonical commutators: $[A_\mu(\vec{x}), A_\nu(\vec{y})] = [\pi^\mu(\vec{x}), \pi^\nu(\vec{y})] = 0$

$$[A_\mu(\vec{x}), \pi_\nu(\vec{y})] = i g_{\mu\nu}\delta^3(\vec{x}-\vec{y})$$

($g_{\mu\nu}$ necessary for Lorentz invariance)

$$\Rightarrow \quad A_\mu(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2|\vec{p}|}}\sum_{\lambda=0}^{3}\varepsilon_\mu^\lambda(\vec{p})\left[a_{\lambda\vec{p}}\, e^{i\vec{p}\vec{x}} + a^+_{\lambda\vec{p}}\, e^{-i\vec{p}\vec{x}}\right]$$

$$\pi^\mu(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{|\vec{p}|}{2}}\,(+i)\sum_{\lambda=0}^{3}\varepsilon_\mu^\lambda(\vec{p})\left[a_{\lambda\vec{p}}\, e^{i\vec{p}\vec{x}} - a^+_{\lambda\vec{p}}\, e^{-i\vec{p}\vec{x}}\right]$$

Compared to the expression of $\pi(\vec{x})$, the conjugate momentum of a scalar field, this has $(+i)$ as overall factor instead of $(-i)$.
Tong points out that this can be traced back to the relation between $\pi^\mu$ and $\dot{A}^\mu$ which follows from the Lagrangian.

$$\pi^\mu \simeq -\dot{A}^\mu + \dots$$

and this implies (in the Heisenberg picture) the $+i$ factor.

Normalization of the polarization vectors (both the choice of the "timelike" vector as the $\varepsilon^0$, as well as the sign of the normalization are arbitrary, but convenient):

$$\varepsilon^\lambda \cdot \varepsilon^{\lambda'} = g^{\lambda\lambda'} \qquad \varepsilon^\lambda \cdot \varepsilon^{\lambda'} = \varepsilon^\lambda_\mu \varepsilon^{\lambda'}_\nu g^{\mu\nu}$$

$$\Rightarrow \quad \varepsilon^\lambda_\mu \varepsilon^{\lambda'}_\nu g_{\lambda\lambda'} = g_{\mu\nu} \qquad \underbrace{\varepsilon^\lambda \cdot \varepsilon^{\lambda'} g_{\lambda\lambda'}}_{g^{\lambda\lambda'} g_{\lambda\lambda'}} = \underbrace{\varepsilon^\lambda_\mu \cdot \varepsilon^{\lambda'}_\nu g_{\lambda\lambda'} g^{\mu\nu}}_{g_{\mu\nu} g^{\mu\nu}}$$

Choice: $\varepsilon^1 \cdot p = \varepsilon^2 \cdot p = 0$ $\qquad$ $\varepsilon^3$ longitudinal

For example for $p \sim (1,0,0,1)$

$$\varepsilon^0 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \varepsilon^1 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad \varepsilon^2 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad \varepsilon^3 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$\left[a_{\lambda,\vec{p}}, a^\dagger_{\lambda',\vec{p}'}\right] = -g_{\lambda\lambda'}(2\pi)^3\delta^3(\vec{p}-\vec{p}')$$

For $\lambda = \lambda' = 0$ the minus sign means that the norm of the states is negative. This is a problem:

$$\langle \vec{p}, 0 | \vec{p}', 0\rangle = -(2\pi)^3\delta^3(\vec{p}-\vec{p}')$$

The solution is to remember the constraint:

$$\partial_\mu A^\mu = 0$$

which we still need to impose.

- This cannot be taken as an operator relation, because $\pi^0 = -\partial_\mu A^\mu$ and this cannot be taken equal to zero, otherwise the canonical commutation relations cannot work.
- One can try to impose this as a condition on the states:

$$\partial_\mu A^\mu |\psi\rangle = 0 \qquad \text{for all the states in our Fock space.}$$

But this is also problematic:

write $\qquad A_\mu(x) = A^+_\mu(x) + A^-_\mu(x)$

with

$$A^{+}_{\mu}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2|\vec{p}|}} \sum_{\lambda=0}^{3} \varepsilon^{\lambda}_{\mu}(\vec{p})\, a_{\lambda,\vec{p}}\, e^{-ipx}$$

$$A^{-}_{\mu}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2|\vec{p}|}} \sum_{\lambda=0}^{3} \varepsilon^{\lambda}_{\mu}(\vec{p})\, a^{+}_{\lambda,\vec{p}}\, e^{+ipx}$$

$A^{+}_{\mu}(x)|0\rangle = 0$ , so $\partial_\mu A^{\mu +}(x)|0\rangle = 0$ as well,

but $\partial_\mu A^{-\mu}(x)|0\rangle \neq 0$ , so we would have to exclude the vacuum from our Fock space, which would make no sense.

This observation points the way to a solution: impose that

$$\partial^\mu A^{+}_{\mu}|\psi\rangle = 0 \qquad \text{Gupta-Bleuler}$$

on all physical states. This also implies that

$$\langle\psi'|\partial_\mu A^\mu|\psi\rangle = 0$$

on all physical states.

Let us try and understand better the meaning of the condition:

$$\partial^\mu A^{+}_{\mu}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2|\vec{p}|}} \sum_{l=0}^{3} \varepsilon^{\lambda}_{\mu}(\vec{p})(-ip^\mu)\, a_{\lambda,\vec{p}}\, e^{-ipx}$$

$$= -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{|\vec{p}|}{2}} \left(a_{0,\vec{p}} - a_{3,\vec{p}}\right) e^{-ipx}$$

So, if we decompose any state in the Fock space as: $|\psi_T\rangle|\phi\rangle$ with $|\psi_T\rangle$ containing only photons created by $a^\dagger_{1,\vec p}$, and $a^\dagger_{2,\vec p}$ i.e. transverse ones, and $|\phi\rangle$ satisfying the condition:

$$\left(a_{0,\vec p} - a_{3,\vec p}\right)|\phi\rangle = 0$$

if $|\phi\rangle$ is a one-particle state, the solution is easy:

$$\left(a_{0,\vec p} - a_{3,\vec p}\right)\left(a^\dagger_{0,\vec p'} - a^\dagger_{3,\vec p'}\right)|0\rangle = -(2\pi)^3\delta^3(\vec p - \vec p')\overbrace{(1-1)}^{0}|0\rangle$$

We can then work by induction, from $(n-1)$- to $n$-particle states:

$$|\phi_n\rangle = \left(a^\dagger_{0,\vec p_n} - a^\dagger_{3,\vec p_n}\right)|\phi_{n-1}\rangle$$

$$\begin{aligned}\left(a_{0,\vec p} - a_{3,\vec p}\right)\left(a^\dagger_{0,\vec p_n} - a^\dagger_{3,\vec p_n}\right)|\phi_{n-1}\rangle &= -(2\pi)^3\delta^3(\vec p - \vec p')\overbrace{(1-1)}^{0}|\phi_{n-1}\rangle + \\ &\quad -\left(a_{0,\vec p}\, a^\dagger_{3,\vec p_n} + a_{3,\vec p}\, a^\dagger_{0,\vec p_n}\right)|\phi_{n-1}\rangle \\ &\quad +\left(a^\dagger_{0,\vec p_n} a_{0,\vec p} + a^\dagger_{3,\vec p_n} a_{3,\vec p}\right)|\phi_{n-1}\rangle \\ &= \left(a^\dagger_{3,\vec p_n} - a^\dagger_{0,\vec p_n}\right)\underbrace{\left(a_{3,\vec p} - a_{0,\vec p}\right)|\phi_{n-1}\rangle}_{\overset{!!}{0}}\end{aligned}$$

by assumption.

So, we started from

$$|\phi_1\rangle = \left(a^\dagger_{0,\vec p_1} - a^\dagger_{3,\vec p_1}\right)|0\rangle$$

and by the argument above we get.

$$|\phi_n\rangle = \prod_{k=1}^{n}\left(a^\dagger_{0,\vec p_k} - a^\dagger_{3,\vec p_k}\right)|0\rangle$$

Clearly, any linear combination: $|\phi\rangle = \sum_n c_n|\phi_n\rangle$ will satisfy the same property.

It is easy to show that $\langle\phi_n|\phi_n\rangle = 0$, by relying on the commutation relation: $\left[a_{3,\vec p} - a_{0,\vec p},\, a^\dagger_{3,\vec p'} - a^\dagger_{0,\vec p'}\right] = 0$.

In general we have $\langle\phi_m|\phi_n\rangle = \delta_{m0}\delta_{n0}$,

so the scalar product in the space of physical states is semi-positive definite.

How do these states contribute to observables? Let us consider the Hamiltonian:

$$H = \int \frac{d^3p}{(2\pi)^3} |\vec{p}| \left( \sum_{i=1}^{3} a^\dagger_{i\vec{p}} a_{i\vec{p}} - a^\dagger_{0\vec{p}} a_{0\vec{p}} \right)$$

Expectation values of the Hamiltonian satisfy:

$$\langle \psi | a_3^\dagger a_3 - a_0^\dagger a_0 | \psi \rangle = \langle \psi | a_3^\dagger (a_3 - a_0) | \psi \rangle = 0$$

so that:

$$\langle \psi | H | \psi \rangle = \int \frac{d^3p}{(2\pi)^3} |\vec{p}| \langle \psi | \left( a^\dagger_{1\vec{p}} a_{1\vec{p}} + a^\dagger_{2\vec{p}} a_{2\vec{p}} \right) | \psi \rangle$$

If $|\psi\rangle = |\psi_T\rangle |\phi\rangle$, the property

$$\langle \phi_m | \phi_n \rangle = \delta_{m0} \delta_{n0}$$

implies that the expectation value of $H$ on $|\psi\rangle$ is identical to the expectation value on $|\psi_T\rangle$.

One can also show that expectation values of any observables satisfy the same property, i.e. that they are independent of the $C_n$ coefficients.

Propagator:

$$\langle 0 | T A_\mu(x) A_\nu(y) | 0 \rangle = \int \frac{d^4p}{(2\pi)^4} \frac{-i g_{\mu\nu}}{p^2 + i\varepsilon} e^{-ip(x-y)}$$

If one modifies the Lagrangian with $-\frac{1}{2\alpha}(\partial_\mu A^\mu)^2$ instead, the propagator becomes:

$$\langle 0 | T A_\mu(x) A_\nu(y) | 0 \rangle = \int \frac{d^4p}{(2\pi)^4} \frac{-i}{p^2 + i\varepsilon} \left( g_{\mu\nu} + (\alpha - 1) \frac{p^\mu p^\nu}{p^2} \right) e^{-ip(x-y)}$$