Quantization of a massive vector field.

Weinberg shows that if you consider the transformation properties of such a field, you can split its four degrees of freedom into:

1: derivative of a scalar field $\partial_\mu \phi$

3: spin-1 vector field with its 3 polarization states.

The latter is also called the Proca field, whose behavior is dictated by the following Lagrangian.

Lagrangian of the Proca field:

$$\mathcal{L}_P = \underbrace{-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \mu^2 A_\rho A^\rho}_{\text{free part}} \underbrace{\left(- J_\mu A^\mu \right)}_{\text{coupling to ext. current.}}$$

EoM.

$$\partial^\mu F_{\mu\nu} + \mu^2 A_\nu = 0 \quad (J_\nu)$$

Take the divergence:

$$\cancel{\partial^\nu \partial^\mu F_{\mu\nu}}^{\,0} + \mu^2 \partial^\nu A_\nu = 0$$

$$\partial^\nu A_\nu = 0 \qquad (\mu \neq 0)$$

$\hookrightarrow$ Klein-Gordon eq.:

$$(\Box + \mu^2) A_\nu = 0$$

Solution

$$A_\mu = \varepsilon_\mu^{(r)}(k)\, e^{-ikx} \qquad r = 1,2,3$$

$$P_{\mu\nu} = \sum_{r=1}^{3} \varepsilon_\mu^{(r)} \cdot \varepsilon_\nu^{(r)*} = -g_{\mu\nu} + \frac{k_\mu k_\nu}{\mu^2}$$

The quantization procedure can be carried out in the standard way.

The problems we encountered when trying to quantize the em. field do not show up if $\mu \neq 0$.

Another way to see that the $\mu \to 0$ limit is problematic is the following: couple A_μ to an external current J^μ and consider the probability to create a vector meson of polarization r by the current.

$$A_r \propto \varepsilon_\mu^{(r)} \cdot J^\mu$$

Four-divergence:

$$\cancel{\partial^\nu \partial^\mu F_{\mu\nu}}^{\,0} + \mu^2 \partial^\nu A_\nu = \partial^\nu J_\nu$$

$$\Rightarrow \quad \partial_\mu A^\mu = \frac{1}{\mu^2} \partial^\mu J_\mu$$

In the massless limit the divergence of A^μ explodes, unless it is coupled to a conserved current $\Rightarrow \partial_\mu J^\mu = 0$ otherwise we cannot take the $\mu \to 0$ limit. This excludes a possible term of the form $k_\mu \cdot J^\mu$.

Let $\vec{k} \propto \hat{z}$: $\quad k^\mu = \left(\sqrt{|\vec{k}|^2 + \mu^2}, 0, 0, |\vec{k}|\right)$

In this frame $\quad \varepsilon^{(1)} = (0,1,0,0) \quad , \quad \varepsilon^{(2)} = (0,0,1,0)$

and $\quad \varepsilon^{(3)} = \frac{1}{\mu}\left(|\vec{k}|, 0, 0, \sqrt{|\vec{k}|^2 + \mu^2}\right)$

so that $\quad \varepsilon^{(r)} \cdot k = 0 \quad ; \quad \varepsilon^{(r)} \cdot \varepsilon^{(s)} = -\delta_{rs}$

The amplitude for creating a meson of polarization $r=3$ is given by:

$$A_3 = \frac{1}{\mu}\left(|\vec{k}| J_0 - \sqrt{|\vec{k}|^2 + \mu^2}\, J_3\right)$$

but $\quad J \cdot k = 0 \to \quad J_0 \sqrt{|\vec{k}|^2 + \mu^2} - |\vec{k}| J_3 = 0 \quad \Rightarrow \quad J_3 = J_0 \frac{\sqrt{|\vec{k}|^2 + \mu^2}}{|\vec{k}|}$

$$\Rightarrow \quad A_3 = \frac{1}{\mu} J_0 \left(|\vec{k}| - \frac{|\vec{k}|^2 + \mu^2}{|\vec{k}|} \right) = -\frac{\mu J_0}{|\vec{k}|} \xrightarrow{\mu \to 0} 0$$

$\varepsilon_\mu^{(\pm)} = \frac{1}{\sqrt{2}} \left(\varepsilon_\mu^{(1)} \pm i \varepsilon_\mu^{(2)} \right)$ are polarization vectors for helicity states ± 1;

$\varepsilon_\mu^{(3)}$ corresponds to helicity 0.

The probability to create a vector meson with helicity 0 is proportional to the mass, and vanishes in the massless limit.

Let us now study the coupling of photons to other fields, in other words, let us consider explicit forms of J_μ as a dynamical object (operator), instead of an external current.

If we couple the photon to other fields, scalars or spinors, we will have to add to the Lagrangian a piece

$$\mathcal{L}'(\phi, \psi, \bar{\psi}, A_\mu)$$

The EoM will be modified as follows:

$$\partial_\nu F^{\nu\mu} + \mu^2 A^\mu + \frac{\delta \mathcal{L}'}{\delta A_\mu} = 0$$

$\frac{\delta \mathcal{L}'}{\delta A_\mu} \equiv J^\mu$ because if $\mathcal{L}'$ is Lorentz-invariant, $\frac{\delta \mathcal{L}'}{\delta A_\mu}$ has to transform like a four-vector. But then we are in the same situation we encountered above, just with a different meaning for J^μ. The conclusion, however, remains: to ensure a sensible $\mu \to 0$ limit, J^μ must be a conserved current.

Gauge invariance.

If, on the other hand, we want to couple the EM-four-potential to matter, we have a different problem, namely to respect gauge inv.

$A_\mu \to A_\mu + \partial_\mu \lambda$ has to leave $\mathcal{L}$ unchanged.

But how can we impose that with $\mathcal{L}'$ the total Lagrangian is gauge invariant? Coleman argues as follows:

$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is g.i.

$$\delta\left(\frac{1}{2}\mu^2 A_\mu A^\mu\right) = \mu^2 A_\mu \partial^\mu \lambda$$

But it's S which has to remain unchanged:

$$S_{tot} = S_P + S'$$

One could, in principle, invent transformation laws for ϕ and ψ such that

$$\delta S' = -\delta S_P ,$$

but if we exclude this possibility, then both $\delta S' = 0 = \delta S_P$

$$\delta S_P = \mu^2 \int d^4x \, A_\mu(x) \partial^\mu \lambda(x) \xrightarrow{p.i.} -\mu^2 \int d^4x \, \partial_\mu A^\mu(x) \lambda(x)$$

$\delta S_P = 0$ only if $\partial_\mu A^\mu(x) = 0$.

So, we concluded that g.i. of both S_P and S' implies $\partial_\mu A^\mu = 0$ for $\mu \neq 0$.

From the EoM we then conclude that

$$\partial_\mu J^\mu = 0$$

$\Rightarrow$ If we address two issues:

- sensible $\mu \to 0$ limit of a theory of a massive vector field;
- gauge invariance if we switch on a mass of the photon;

we come to the same solution $\Rightarrow$ $\partial_\mu J^\mu = 0$

$$\text{where} \quad J^\mu = \frac{\delta \mathcal{L}'}{\delta A_\mu} \quad -$$

Minimal coupling:

Prescription for making a Lagrangian $\mathcal{L}'$ gauge invariant:

replace all four-derivatives by

$$D_\mu = \partial_\mu + ie A_\mu Q$$

where Q defines the variation of the "matter" field under a gauge transformation:

$$\delta \phi = -i Q \phi \, \delta\lambda$$

Examples.

Spinor $\quad \psi \to e^{-ie\lambda(x)} \psi$

$$Q = 1. \quad \Rightarrow \quad D_\mu \psi = (\partial_\mu + ie A_\mu)\psi$$

$$\bar\psi i \partial\!\!\!/ \psi \to i \bar\psi D\!\!\!\!/ \psi = i\bar\psi \partial\!\!\!/ \psi - e A_\mu \bar\psi \gamma^\mu \psi = i \bar\psi \partial\!\!\!/ \psi - e J^\mu A_\mu$$

Scalar : $\quad \partial_\mu \phi^* \partial^\mu \phi \to D_\mu \phi^* D^\mu \phi = (\partial_\mu - ie A_\mu)\phi^* (\partial^\mu + ie A^\mu)\phi$

$$\mathcal{L}_I = -ie \left[\phi^* \partial^\mu \phi - \partial^\mu \phi^* \cdot \phi\right] A_\mu + e^2 A_\mu A^\mu \phi^* \phi$$

$$J^\mu = -\frac{1}{e} \frac{\partial \mathcal{L}_I}{\partial A_\mu} = ie\left[\phi^* \partial^\mu \phi - \partial^\mu \phi^* \cdot \phi\right] - 2e A_\mu \phi^* \phi$$

The minimal-coupling argument can also be introduced as follows:

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \quad \text{is invariant under } e^{-i\alpha}\psi$$

Does a global transformation make sense in a setting where causality is imposed taking into account the finite speed of light?

Local transformations seem more appropriate:

$$\alpha = e\lambda(x)$$

$$\mathcal{L}' = \bar{\psi} e^{ie\lambda(x)} (i\partial\!\!\!/ - m) e^{-ie\lambda(x)} \psi$$

$$= \bar{\psi}(i\partial\!\!\!/ - m)\psi + e\bar{\psi}\,\partial\!\!\!/\lambda\,\psi$$

$\mathcal{L}' = \mathcal{L}$ can be achieved by replacing ∂_μ with

$$D_\mu = \partial_\mu + ieA_\mu$$

$$\text{with } A_\mu \to A_\mu + \partial_\mu \lambda$$

Same argument for

$$\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$$

$\phi \to e^{-i\alpha}\phi$ is a symmetry, and if we change it to a local transformation we get

$$\partial_\mu (e^{ie\lambda(x)} \phi^*) \partial^\mu (e^{-ie\lambda(x)} \phi) - m^2 \phi^* \phi,$$

which is not invariant. $\partial_\mu \to D_\mu + ieA_\mu$ does the trick.

$$\phi' = e^{-ie\lambda(x)} \phi(x)$$

$$(D_\mu \phi)' = e^{-ie\lambda(x)} D_\mu \phi(x).$$

covariant derivative.

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi$$

$$\mathcal{L}_{SQED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \phi^* D^\mu \phi - M^2 \phi^* \phi$$

The derivation of the Feynman rules for these two theories and the calculation of some simple processes at tree level are part of the exercise series of this week.