Gaussian integrals

1D:

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}}$$

nD: define $(x, Ax) = x_a A_{ab} x_b$ for A a symmetric matrix.

Bring A in diagonal form with an orthogonal transf.

then : $(\vec{x}, A\vec{x}) = \sum_a \lambda_a x_a^2$ with λ_a the eigenvalues of A

$$A \xrightarrow{\text{diag}} \begin{pmatrix} \lambda_1 & \cdots & 0 \\ 0 & & \lambda_n \end{pmatrix}$$

Define $[dx] = \frac{d^n x}{(2\pi)^{n/2}}$, then we have

$$\int [dx]\, e^{-\frac{1}{2}(\vec{x}, A\vec{x})} = \left(\prod_n \lambda_n\right)^{-1/2} = (\det A)^{-1/2}$$

Consider now a general quadratic form:

$$Q(\vec{x}) = \frac{1}{2}(\vec{x}, A\vec{x}) + (\vec{b}, \vec{x}) + c$$

at $\vec{x}_0 \equiv -A^{-1}\vec{b}$, Q is minimal:

$$Q(\vec{x}_0) = \frac{1}{2}(\vec{b}, A^{-1}\vec{b}) - (\vec{b}, A^{-1}\vec{b}) + c = -\frac{1}{2}(\vec{b}, A^{-1}\vec{b}) + c$$

$$\vec{Q}(\vec{x}) = Q(\vec{x}_0)$$

$$\frac{1}{2}\left(\vec{x} - \vec{x}_0 + \vec{x}_0, A(\vec{x} - \vec{x}_0 + \vec{x}_0)\right) + (\vec{b}, \vec{x} - \vec{x}_0) + (\vec{b}, \vec{x}_0) + c =$$

$$= \frac{1}{2}(\vec{x} - \vec{x}_0, A(\vec{x} - \vec{x}_0)) + \underbrace{(\vec{x}_0, A(\vec{x} - \vec{x}_0))}_{-\cancel{(\vec{b}, \vec{x} - \vec{x}_0)}} + \cancel{(\vec{b}, \vec{x} - \vec{x}_0)} + Q(\vec{x}_0)$$

$$= \frac{1}{2}(\vec{x} - \vec{x}_0, A(\vec{x} - \vec{x}_0)) + Q(\vec{x}_0)$$

so that $$\int [dx]\, e^{-\frac{1}{2}(\vec{x}, A\vec{x})} = e^{-Q(\vec{x}_0)} \int [dy]\, e^{-\frac{1}{2}(\vec{y}, A\vec{y})}$$
$$= e^{-Q(\vec{x}_0)} \left(\det A\right)^{-1/2}$$

Let's now introduce polynomials in the integral.

$$\int [dx]\, P(\vec{x})\, e^{-Q(\vec{x})} = \int [dx]\, P\left(-\frac{\partial}{\partial \vec{b}}\right) e^{-Q(x)}$$
$$= P\left(-\frac{\partial}{\partial \vec{b}}\right) e^{-\tilde{Q}(\vec{x}_0)} \cdot \left(\det A\right)^{-1/2}$$

And finally integrals over complex variables:

define $\vec{z} = \frac{1}{\sqrt{2}}(\vec{x} + i\vec{y})$, then we have

$$[dz^*][dz] = [dx][dy]$$

and from this:

$$\int [dz^*][dz]\, e^{-(\vec{z}^*, A\vec{z})} = \left(\det A\right)^{-1}$$

since the integrals over x and over y factorize, are identical and each give the square root of $\det A^{-1}$

All this is the basis of the path-integral formulation of QM, even though one still needs to assume that the step from finite- to infinite-dimensional vector spaces over which one is integrating, can be performed without changes and does not introduce problems.

If we have a space of functions with inner product

$$(\phi_1, \phi_2) = \int d^4x\, \phi_1(x)\, \phi_2(x)$$

then we can define quadratic forms as:

$$Q[\phi] = \frac{1}{2}\int d^4x\, d^4y\, \phi(x) A(x,y) \phi(y) + \int d^4x\, b(x)\phi(x) + c$$

If we have a discrete basis $\varphi_i(x)$, then

$$\phi(x) = \sum_i c_i \varphi_i(x)$$

and I can transform $A(x,y)$ into a matrix A_{ij} and $b(x)$ into a vector b_i and write the quadratic form as before, even though matrices and vectors will now be infinitely long.

Path integrals in field theory.

$$\mathcal{L}(\phi, J) = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 + \mathcal{L}_I(\phi) + J\phi$$

$$S_c[\phi, J] = \int d^4x\, \mathcal{L}(\phi, J)$$

Statement: the path integral of $e^{iS_c[\phi,J]}$ is the generating functional of all Green's functions.

$$Z[J] = N \int [d\phi]\, e^{iS_c[\phi,J]}.$$

$Z[J] = \langle 0|S|0\rangle_J$ with S the S-matrix, and J as subscript indicate that the vacuum refers to an Hamiltonian where the term $-\phi J$ has been added.

We will show later that $Z[J]$ contains all Green's functions of the theory, namely:

$$Z[J] = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \cdots d^4x_n \, G^{(n)}(x_1, \ldots, x_n) J(x_1) \cdots J(x_n)$$

where $G^{(n)}$ are equal to

$$G^{(n)}(x_1, \ldots, x_n) = \langle 0 | T \phi(x_1) \cdots \phi(x_n) | 0 \rangle$$

From $Z[J]$ one can derive all Green's functions by differentiating with respect to $J(x)$:

$$G^{(n)}(x_1, \ldots, x_n) = i^n \left. \frac{\delta Z[J]}{\delta J(x_1) \cdots \delta J(x_n)} \right|_{J(x)=0}$$

I also state without proof that the logarithm of $Z[J]$ is the generating functional of all connected Green's functions:

$$Z[J] = \exp[iW[J]]$$

$$iW[J] = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \cdots d^4x_n \, G_c^{(n)}(x_1, \ldots, x_n) J(x_1) \cdots J(x_n)$$

with $G_c^{(n)}$ the <u>connected</u> Green's functions.

How can we give meaning, or attempt to calculate the integral? First of all, the exponential will be an oscillating function if S_c is real. If we want to relate it to the Gaussian integrals we have discussed we need first to do a Wick rotation:

$$x_0 = -i x_4$$

$$dx_0 d^3x = -i d^4x_E$$

$$\Rightarrow \quad i \int d^4x \, \mathcal{L}(\phi, J) = \int d^4x_E \, L(\phi, J) = -S_E[\phi, J]$$

The reason for defining $S_E = -\int d^4x_E \, L$ is that:

$$\frac{1}{2}\left(\partial_\mu\phi\,\partial^\mu\phi - m^2\phi^2\right) \longrightarrow -\frac{1}{2}\overbrace{\left(\partial_\mu\phi\,\partial_\mu\phi + m^2\phi^2\right)}^{\text{positive definite}}$$

because with our metric $\partial_\mu\phi\,\partial^\mu\phi = (\partial_0\phi)^2 - (\vec{\nabla}\phi)^2 \longrightarrow -\left[(\partial_4\phi)^2 + (\vec{\nabla}\phi)^2\right]$.

With this choice the exponential has a negative argument and we can rely on Gaussian integration.

Path integral for a free theory.

$$iS_c[\phi, J] = \frac{i}{2}\int d^4x \left[(\partial_0\phi)^2 - (\vec{\nabla}\phi)^2 - m^2\phi^2 + J\phi \right]$$

$$= -\frac{1}{2}\int d^4x_E \left[\phi(-\Box_E + m^2)\phi - J\phi\right] \qquad \text{where } \Box_E = \partial_\mu\partial_\mu$$

$$\int [d\phi] \exp\left\{-\frac{1}{2}(\phi, A\phi) + (b, \phi)\right\} = \exp\left[\frac{1}{2}(b, A^{-1}b)\right](\det A)^{-1/2}$$

$$A = -\Box_E + m^2 \quad ; \quad b = J$$

$$A^{-1} = \frac{1}{-\Box_E + m^2}$$

$$Z_E[J] = N \det(-\Box_E + m^2)^{-1/2} \exp\left\{\frac{1}{2}\int d^4x_E \, J(x) \frac{1}{-\Box_E + m^2} J(x)\right\}$$

$$= \exp\left\{\frac{1}{2}\int d^4x_E \, J(x) \frac{1}{-\Box_E + m^2} J(x)\right\}$$

after imposing the condition $Z_E[0] = 1$.

One can be more explicit and write:

$$\int d^4x_E \, J(x) \frac{1}{-\Box_E + m^2} J(x) = \int d^4x_E \, d^4y_E \, J(x) \, i\Delta_E(x-y) \, J(y)$$

with $\left(-\Box_E + m^2\right) \Delta_E(x-y) = -i \, \delta^4(x-y)$

By Fourier transformation:

$$\Delta_E(x) = -i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik_E \cdot x_E}}{k_E^2 + m^2}$$

Performing the Wick rotation backward we get:

$$Z_0[J] = \exp\left\{ -\frac{1}{2} \int d^4x \int d^4y \, J(x) \, \Delta_F(x-y) \, J(y) \right\}$$

with Δ_F the Feynman propagator.

<u>Path integral for an interacting field theory.</u>

If we have an interaction term in the Lagrangian, this will contain higher powers than the second in the fields and will bring us beyond Gauss integrals. However, as long as we are ready to expand the exponential with the interaction term, we can resort to the formula for the integral of polynomials times Gaussian.

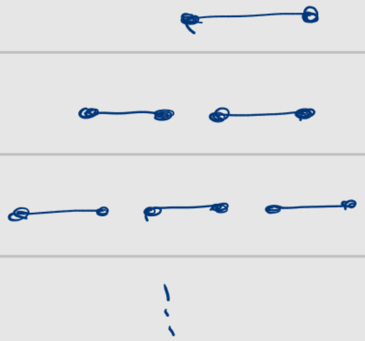
$$Z[J] = N \int [d\phi] \exp\left\{ i \int d^4x \left(\mathcal{L}_0(\phi) + \mathcal{L}_I(\phi) + \phi J \right) \right\}$$

$$= N \int [d\phi] \left[\sum_{n=0}^{\infty} \frac{i^n}{n!} \left(\int d^4x \, \mathcal{L}_I(\phi) \right)^n \right] \exp\left\{ i \int d^4x \left(\mathcal{L}_0(\phi) + \phi J \right) \right\}$$

$$= N \left[\sum_{n=0}^{\infty} \frac{i^n}{n!} \left(\int d^4x \, \mathcal{L}_I\left(\frac{-i\delta}{\delta J}\right) \right)^n \right] Z_0[J]$$

Describe how one gets the piece:

at fourth order in J from the first order in the interaction Lagrangian acting on the fourth-order term in the expansion of $Z_0[J]$. The latter can be represented by



1st order

2nd order

3rd order

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