Path integral quantization: no operators. Expectation values obtained as averages of functions with an integral measure determined by the Lagrangian. How to translate anticommutation rules for operators into an analogous property for functions?

$\Rightarrow$ anticommuting c-numbers = Grassmann variables.

Be $\eta, \xi, \bar{\eta}, \bar{\xi}$ Grassmann variables (bar = complex conjugation)

then we have

$$\{\eta, \xi\} = 0 \quad ; \quad \{\bar{\eta}, \bar{\xi}\} = 0 \quad , \dots$$

$$\Rightarrow \quad \eta^2 = \bar{\eta}^2 = \xi^2 = \bar{\xi}^2 = 0$$

Functions are defined via their Taylor expansion, which in this case simplifies considerably: consider, for example

$$e^{\eta} = 1 + \eta + \frac{1}{2}\eta^2 + \dots = 1 + \eta$$

Can we calculate integrals over Grassmann variables?

Let us first impose some properties of integrals and then determine their value in a way which is compatible with these properties:

1. Linearity

$$\int d\eta \left[\alpha F_1(\eta) + \beta F_2(\eta)\right] = \pm \alpha \int d\eta\, F_1(\eta) \pm \beta \int d\eta\, F_2(\eta)$$

where the minus sign applies if α and β are also Grassmann variables.

If we consider a double integral, we get instead:

$$\int d\eta\, d\bar\eta \left[\alpha F_1(\eta,\bar\eta) + \beta F_2(\eta,\bar\eta) \right] = \alpha \int d\eta\, d\bar\eta\, F_1(\eta,\bar\eta) + \beta \int d\eta\, d\bar\eta\, F_2(\eta,\bar\eta)$$

2. Translation invariance

$$\int d\eta\, F(\eta) = \int d\eta\, F(\eta+\xi)$$

with ξ another Grassmann variable.

3. Normalization:

$$\int d\eta\, d\bar\eta\, e^{\bar\eta\eta} = 1$$

Since the exponential as well as any other function can only be linear in its argument, this is a universal condition which fixes all integrals. Let's see how:

$$\int d\eta\, d\bar\eta\, f(\eta,\bar\eta)$$

$f(\eta,\bar\eta) = a + b\eta + c\bar\eta + d\bar\eta\eta$, so the integral will be a linear combination of the four coefficients $a, \dots, d$.

Consider translation invariance:

$$\int d\eta\, g(\eta) = \int d\eta\, g(\eta+\xi)$$

$$\underbrace{\int d\eta\,(A+B\eta)}_{= A\int d\eta + B\int d\eta\,\eta} = \underbrace{\int d\eta\,(A+B\xi+B\eta)}_{A\int d\eta + B\int d\eta\,\eta \;\underbrace{- B\xi\int d\eta}_{=0}}$$

So, we conclude that

$$\int d\eta = 0 \implies \int d\bar\eta = 0$$

From which it follows that:

$$\int d\eta\, d\bar\eta = 0 \qquad \int d\eta\, d\bar\eta\, \eta = \int d\eta\, d\bar\eta\, \bar\eta = 0$$

$$\int d\eta\, d\bar\eta\, e^{\bar\eta\eta} = \int d\eta\, d\bar\eta\,(1+\bar\eta\eta) = \int d\eta\, d\bar\eta\, \bar\eta\eta = \left[\int d\eta\,\eta\right]\left[\int d\bar\eta\,\bar\eta\right]$$

Our normalization corresponds to setting

$$\int d\eta\, \eta = \int d\bar\eta\, \bar\eta = 1$$

More in general we have

$$\int d\eta\, d\bar\eta\, (1, \eta, \bar\eta, \bar\eta\eta) = (0, 0, 0, 1)$$

so integration of a Grassmann variable works exactly as differentiation.

We can now introduce a constant in the exponential:

$$\int d\eta\, d\bar\eta\, e^{a\bar\eta\eta} = \int d\eta\, d\bar\eta\,(1+a\bar\eta\eta) = a$$

Note that with commuting numbers we have:

$$\int dz\, dz^*\, e^{-a z^* z} = \frac{2\pi}{a}$$

$$z = \frac{x+iy}{\sqrt2} \qquad z^* z = \frac{x^2+y^2}{2} \quad \Rightarrow \quad \int dz\, dz^*\, e^{-a z^* z} = \left(\int dx\, e^{-\frac12 a x^2}\right)^2 = \frac{2\pi}{a}$$

If we now extend the case to n Grassmann variables and a general quadratic form:

$$(\bar\eta, A\eta) = \sum_{i,j} \bar\eta_i A_{ij} \eta_j = \sum_i \bar\eta'_i \eta'_i a_i$$

where a_i, $i=1,\dots,n$ are the eigenvalues of the matrix A.

We then have for a $2n$-dimensional integral:

$$\int (d\eta)(d\bar\eta)\, e^{(\bar\eta, A\eta)} = \int (d\eta)(d\bar\eta) \prod_r e^{\bar\eta_r \eta_r a_r} = \prod_r a_r = \det A$$

where $\quad (d\eta)(d\bar\eta) = d\eta_1 d\bar\eta_1 \cdots d\eta_n d\bar\eta_n$

We will now extend this to integrals over infinite-dimensional linear spaces:

$$\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix} \quad , \quad \bar\psi = (\bar\psi_1, \dots, \bar\psi_N)$$

Consider an action of the form:

$$S = \int d^4x \left[\bar\psi A(\phi) \psi\right] + S_B(\phi) = (\bar\psi, A\psi) + S_B[\phi(x)]$$

where $\phi(x)$ is a bosonic field. We could have for example:

$$\bar\psi(x) A(x) \psi(x) = \bar\psi(x)\left(i\not\partial - m + i\gamma_5 \phi(x)\right)\psi(x)$$

Following the same generalization we made for scalar fields we would now conclude that:

$$\int [d\psi][d\bar\psi]\, e^{iS} \propto \det A$$

Can we understand this? Like in the scalar case we would want to have:

$$\langle 0|S|0\rangle = N \int [d\psi][d\bar\psi]\, e^{iS}$$

What is $\quad \langle 0|S|0\rangle$? From a Feynman-diagram perspective it is the exponential of the sum of all connected vacuum-to-vacuum diagrams:

$$\langle 0|S|0\rangle = \exp\Big\{-\Big[$$

$$+ \cdots \Big]\Big\}$$

where the minus sign comes from the fermion loop.

If we consider the same path integral with N bosonic fields $\phi_1, \dots, \phi_N$,

i.e. $$S = \int d^4x\, \phi^*(x) A(\varphi) \phi(x)$$

we would come to the conclusion that

$$N \int [d\phi][d\phi^*]\, e^{iS} = (\det A)^{-1}$$

which would also correspond to the vacuum-to-vacuum amplitude

$$(\det A)^{-1} = \langle 0|S|0\rangle = \exp\left\{ \left[\bigcirc + \bigcirc + \bigcirc + \dots \right] \right\}$$

note that in this case we do not have a minus sign because the loop is of a bosonic particle.

So, the picture is consistent: a quadratic action gives the determinant for fermions and the inverse of the determinant for bosons and the relative sign in the exponential can be understood as originating from the minus sign connected with fermion loops if one thinks of the result in terms of vacuum-to-vacuum amplitudes.

Let us now try to understand in detail the representation in terms of vacuum-to-vacuum diagrams. For this, we need to introduce the GF for fermions after we have introduced appropriate sources for the fermionic fields:

$$Z[\eta, \bar\eta] = N \int [d\psi][d\bar\psi]\, \exp\left\{ i \int d^4x \left[\bar\psi (i\not\partial - m)\psi + \bar\eta\psi + \bar\psi\eta \right] \right\}$$

As in the case of scalars, an explicit expression can be obtained by completing the square, namely

$$Z[\eta,\bar\eta] = \exp\left[-\int d^4x\, d^4y\; \bar\eta(x) S_F(x-y)\eta(y)\right]$$

To obtain correlation functions we need to take derivatives wrt. the sources η and $\bar\eta$. Let us be careful about signs first:

$$\frac{d}{d\eta}\theta\eta = \frac{d}{d\eta}(-\eta\theta) = -\theta$$

which leads us to

$$\langle 0|T\psi(x_1)\bar\psi(x_2)|0\rangle = \left(\frac{1}{i}\frac{\delta}{\delta\bar\eta(x_1)}\right)\left(-\frac{1}{i}\frac{\delta}{\delta\eta(x_2)}\right) Z[\eta,\bar\eta]\Big|_{\eta=\bar\eta=0}$$

$$= S_F(x_1-x_2)$$

We can then argue that

$$\det A = \left\{\exp\left[-\int d^4x\, \frac{\delta}{\delta\eta(x)}\, \gamma_5\phi(x)\frac{\delta}{\delta\bar\eta(x)}\right] Z[\eta,\bar\eta]\right\}\Big|_{\eta=\bar\eta=0}$$

which illustrates how the sum of all one-loop diagrams written above in the argument of the exponential comes about.

Quantization of the em. field.

Functional integral for the free action of the em. field:

$$\int [dA]\, e^{iS[A]}$$

with $S[A] = \int d^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right] = \frac{1}{2}\int d^4x\, A_\mu(x)\left[\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu\right]A_\nu(x)$

$$= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\, \tilde{A}_\mu(k)\left(-k^2 g^{\mu\nu} + k^\mu k^\nu\right)\tilde{A}_\nu(-k)$$

All field configurations of the kind $\tilde{A}_\mu(k) = k_\mu \alpha(k)$ with α a generic scalar function, give $S=0 \Rightarrow \exp(iS)=1$. Integrating over all these field configurations gives an infinite constant factor in the path integral. How can we get rid of this factor? The solution is due to Faddeev and Popov (1967). The idea is the following: if we wish to impose a gauge we have to integrate only over field configurations which satisfy the gauge condition of choice. For example, if we consider the Lorentz gauge

$$\partial_\mu A^\mu = 0$$

we want to integrate only over field configurations which fulfill this constraint. We then define

$$G(A) = \partial_\mu A^\mu$$

and will insert in the integral a δ-function $\delta(G(A))$ that will force us to consider only these field configurations. Of course we need to introduce the δ-function in a way which gives a finite integral rather than infinity. To do so we insert the δ-function together with an integral over an auxiliary field:

$$1 = \int [d\alpha]\, \delta(G(A^\alpha)) \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)$$

where $\quad A^\alpha_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x)$

This is a generalization of the identity

$$1 = \left(\prod_i \int da_i\right) \delta^{(n)}(\vec{g}(\vec{a})) \det\left(\frac{\partial g_i}{\partial a_j}\right).$$

What is $\quad \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)$?

For $G(A^\alpha) = \partial^\mu A_\mu + \frac{1}{e}\partial^2\alpha \qquad \frac{\delta G(A^\alpha)}{\delta\alpha} = \frac{1}{e}\partial^2$ which is indep. of A_μ.

Let's go back to the original functional integral:

$$\int [dA]\, e^{iS[A]} = \det\frac{\delta G(A^\alpha)}{\delta\alpha} \int [d\alpha] \int [dA]\, e^{iS[A]}\, \delta(G(A^\alpha))$$

We change variables from A to A^α and get:

$$\det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) \int [d\alpha] \int [dA]\, e^{iS[A]} \delta(G(A))$$

Let us now generalize the gauge to $\quad G(A) = \partial^\mu A_\mu(x) - \omega(x)$ which doesn't change the determinant or any of the above steps.

$$\det\left(\frac{1}{e}\partial^2\right)\left(\int [d\alpha]\right) \int [dA]\, e^{iS[A]}\, \delta(\partial^\mu A_\mu - \omega(x))$$

Since this expression is valid for any $\omega(x)$, we can also integrate over all possible functions with $N(\xi)\exp\left[-i\int d^4x\, \frac{\omega^2(x)}{2\xi}\right]$ as weight function.

So we end up with:

$$N(\xi)\,\det\left(\frac{\partial^2}{e}\right)\left(\int[d\alpha]\right)\int[dA]\,e^{iS[A]}\underbrace{\int[d\omega]\,e^{-i\int d^4x\,\frac{\omega^2(x)}{2\xi}}\,\delta(\partial^\mu A_\mu-\omega(x))}_{e^{-i\int d^4x\,\frac{(\partial^\mu A_\mu)^2}{2\xi}}}$$

$$\Rightarrow\quad S[A]\longrightarrow\int d^4x\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{1}{2\xi}(\partial^\mu A_\mu)^2\right]$$

When we started, we obtain for $S[A]$ the expression:

$$\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\,\tilde{A}_\mu(k)\left(-k^2g^{\mu\nu}+k^\mu k^\nu\right)\tilde{A}_\nu(-k)$$

with a kernel function in k-space which is not invertible:

the $-k^2g^{\mu\nu}+k^\mu k^\nu$ is singular as 4×4 matrix (has zero eigenvalues).

After implementing the FP trick we end up with:

$$-k^2g^{\mu\nu}+\left(1-\frac{1}{\xi}\right)k^\mu k^\nu$$

which is not singular anymore and can be inverted:

$$\left(-k^2g_{\mu\nu}+\left(1-\frac{1}{\xi}\right)k_\mu k_\nu\right)\tilde{D}_F^{\nu\rho}(k)=i\,\delta_\mu{}^\rho$$

$$\tilde{D}_F^{\mu\nu}(k)=\frac{-i}{k^2+i\varepsilon}\left(g^{\mu\nu}-(1-\xi)\frac{k^\mu k^\nu}{k^2}\right)$$

Check:

$$\left(-k^2g_{\mu\nu}+\left(1-\frac{1}{\xi}\right)k_\mu k_\nu\right)\frac{1}{k^2}\left(g^{\nu\rho}-(1-\xi)\frac{k^\nu k^\rho}{k^2}\right)=$$

$$=-\delta_\mu{}^\rho+(1-\xi)\frac{k_\mu k^\rho}{k^2}+\left(1-\frac{1}{\xi}\right)\frac{k_\mu k^\rho}{k^2}-\left(1-\frac{1}{\xi}\right)(1-\xi)\frac{k_\mu k^\rho}{k^2}$$

$$=-\delta_\mu{}^\rho+\left(\cancel{1}\cancel{-\xi}+\cancel{1}\cancel{-\frac{1}{\xi}}\cancel{-1}+\cancel{\frac{1}{\xi}}+\cancel{\xi}\cancel{-1}\right)\frac{k_\mu k^\rho}{k^2}=-\delta_\mu{}^\rho\qquad\checkmark$$