Quantum Field Theory I — Exercise 1

Canonical Quantization — 26. 09. 2023

1. Consider a generic scalar field theory, with n independent fields $\phi_a(x)$, quantized imposing the following (equal-time) commutation relations (attention! in this exercise we keep track of $\hbar$ and c):

$$[\phi_a(x), \pi_b(y)] = i\hbar\delta_{ab}\delta^{(3)}(\vec{x} - \vec{y}),$$
$$[\phi_a(x), \phi_b(y)] = [\pi_a(x), \pi_b(y)] = 0.$$

   (a) Show that the momentum operator

$$P^i = \int d^3x \, \pi_a(x) \frac{\partial \phi_a(x)}{\partial x_i} \quad (i = 1, 2, 3),$$

   satisfies the following commutation relations

$$[P^i, \phi_a(x)] = -i\hbar \frac{\partial \phi_a(x)}{\partial x_i}, \quad [P^i, \pi_a(x)] = -i\hbar \frac{\partial \pi_a(x)}{\partial x_i}.$$

   (b) Using the above result, show that any operator $\hat{F}(x) = \hat{F}(\phi_a(x), \pi_a(x))$, which can be expanded as a power series in $\phi_a(x)$ and $\pi_a(x)$, satisfies

$$[P^i, \hat{F}(x)] = -i\hbar \frac{\partial \hat{F}(x)}{\partial x_i}.$$

   Taking into account the time-evolution of the operator (in the Heisenberg picture), show that the equations of motion of the operator can be written in the following manifestly covariant form

$$[P^\mu, \hat{F}(x)] = -i\hbar \frac{\partial \hat{F}(x)}{\partial x_\mu},$$

   where $P^0 = c^{-1}H$.

2. [**2.2** from Peskin & Schröder]

   Consider the field theory of a complex-valued scalar field obeying the Klein-Gordon equation. The action of this theory is

$$S = \int d^4x \, \left(\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \right).$$

   It is easiest to analyse this theory by considering $\phi(x)$ and $\phi^*(x)$, rather than the real and imaginary parts of $\phi(x)$, as the basic dynamical variables.

(a) Find the conjugate momenta to $\phi(x)$ and $\phi^*(x)$ and the canonical commutation relations. Show that the Hamiltonian is

$$H = \int d^3x \left(\pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi\right).$$

Compute the Heisenberg equation of motion for $\phi(x)$ and show that it is indeed the Klein-Gordon equation.

(b) Diagonalize H by introducing creation and annihilation operators. Show that the theory contains two sets of particles of mass m.

(c) Rewrite the conserved charge

$$Q = \int d^3x \, \frac{i}{2} \left(\phi^*\pi^* - \pi\phi\right)$$

in terms of creation and annihilation operators, and evaluate the charge of the particles of each type.

(d) Consider the case of two complex Klein-Gordon fields with the same mass. Label the fields as $\phi_a(x)$, where $a = 1, 2$. Show that there are now four conserved charges, one given by the generalization of part (c), and the other three given by

$$Q^i = \int d^3x \, \frac{i}{2} \left(\phi_a^*(\sigma^i)_{ab}\pi_b^* - \pi_a(\sigma^i)_{ab}\phi_b\right),$$

where σ^i are the Pauli sigma matrices. Show that these three charges have the commutation relations of angular momentum ($SU(2)$). Generalize these results to the case of n identical complex scalar fields.

3. [**2.3** from Peskin & Schröder]

Evaluate the function

$$\langle 0|\phi(x)\phi(y)|0\rangle = D(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}} e^{-ip\cdot(x-y)},$$

for $(x-y)$ spacelike so that $(x-y)^2 = -r^2$, explicitly in terms of Bessel functions.

4. [**1.2** from Coleman]

The time-ordered product of two fields, $A(x)$ and $B(y)$, is defined by

$$T(A(x)B(y)) = \begin{cases} A(x)B(y) & \text{if } \; x_0 > y_0 \\ B(y)A(x) & \text{if } \; y_0 > x_0 \end{cases}$$

Using only the field equation and the equal time commutation relations, show that, for a free scalar field of mass m,

$$\left(\Box_x + m^2\right)\langle 0|T(\phi(x)\phi(y))|0\rangle = c \cdot \delta^{(4)}(x-y)$$

and find c, the constant of proportionality.

5. [**1.3** from Coleman]

Show that

$$\langle 0|T(\phi(x)\phi(y))|0\rangle = \lim_{\epsilon\to 0^+} \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{-c}{p^2 - m^2 + i\epsilon}$$

The limit symbol indicates that ϵ goes to zero from above, i.e., through positive values. (If ϵ was not present, the integral would be ill-defined, because it would have poles in the domain of integration.)

Hint: Do the p_0 *integration first, and compare your result with the expression for the left-hand side obtained by inserting the explicit form of the field* ϕ. *Treat the cases* $x_0 > y_0$ *and* $x_0 < y_0$ *separately.*