Quantum Field Theory I — Exercise 3

Majorana Fermions — 06. 11. 2023

[**3.4** from Peskin & Schröder]

1. We have seen in the lecture that one can write the Dirac equation for Weyl spinors $\psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$ as

$$i\overline{\sigma}^\mu \partial_\mu u_+ = 0,$$
$$i\sigma^\mu \partial_\mu u_- = 0.$$

We also discussed in the lecture how the different components $u_\pm$ transform under Lorentz transformations.

One can write a relativistic equation for a massless 2 -component fermion field that transforms as the upper two components of a Dirac spinor, i.e. like u_+. We will call such a 2-component field $\chi(x)$.

(a) Show that it is possible to write an equation for $\chi(x)$ as a massive field in the following way:

$$i\overline{\sigma} \cdot \partial\chi - im\sigma^2\chi^* = 0.$$

That is, show, first, that this equation is relativistically invariant and, second, that it implies the Klein-Gordon equation, $(\partial^2 + m^2)\chi = 0$. This form of the fermion mass is called a Majorana mass term.

(b) Does the Majorana equation follow from a Lagrangian? The mass term would seem to be the variation of $(\sigma^2)_{ab}\chi_a^*\chi_b^*$; however, since σ^2 is antisymmetric, this expression would vanish if $\chi(x)$ were an ordinary c-number field. When we go to quantum field theory, we know that $\chi(x)$ will become an anticommuting quantum field. Therefore, it makes sense to develop its classical theory by considering $\chi(x)$ as a classical anticommuting field, that is, as a field that takes as values *Grassmann numbers* which satisfy (remember the theoretical exercises!)

$$\alpha\beta = -\beta\alpha \quad \text{for any } \alpha, \beta.$$

Note that this relation implies that $\alpha^2 = 0$. A Grassmann field $\xi(x)$ can be expanded in a basis of functions as

$$\xi(x) = \sum_n \alpha_n \phi_n(x),$$

where the $\phi_n(x)$ are orthogonal c-number functions and the α_n are a set of independent Grassmann numbers. Define the complex conjugate of a product of Grassmann numbers to reverse the order:

$$(\alpha\beta)^* \equiv \beta^*\alpha^* = -\alpha^*\beta^*.$$

This rule imitates the Hermitian conjugation of quantum fields. Show that the classical action,

$$S = \int d^4x \left[\chi^\dagger i\overline{\sigma} \cdot \partial\chi + \frac{im}{2} \left(\chi^T \sigma^2 \chi - \chi^\dagger \sigma^2 \chi^* \right) \right],$$

with $\chi^\dagger = (\chi^*)^T$ is real, i.e. $S^* = S$, and that varying this S with respect to χ and χ^* yields the Majorana equation.

(c) Let us write a 4-component Dirac field as

$$\psi(x) = \begin{pmatrix} u_+ \\ u_- \end{pmatrix},$$

and recall that the lower components of ψ transform in a way equivalent by a unitary transformation to the complex conjugate of the representation u_+. In this way, we can rewrite the 4-component Dirac field in terms of two 2-component spinors:

$$u_+(x) = \chi_1(x), \qquad u_-(x) = i\sigma^2\chi_2^*(x).$$

Rewrite the Dirac Lagrangian in terms of $\chi_1(x)$ and $\chi_2(x)$ and note the form of the mass term.

(d) Show that the action of part (c) has a global symmetry. Compute the divergences of the currents

$$J^\mu = \chi^\dagger \overline{\sigma}^\mu \chi, \quad J^\mu = \chi_1^\dagger \overline{\sigma}^\mu \chi_1 - \chi_2^\dagger \overline{\sigma}^\mu \chi_2,$$

for the theories of parts (b) and (c), respectively, and relate your results to the symmetries of these theories. Construct a theory of N free massive 2-component fermion fields with $O(N)$ symmetry (that is, the symmetry of rotations in an N-dimensional space).