Quantum Field Theory I — Exercise 4

Quantization of the Dirac field and Supersymmetry — 13. 11. 2023

1. [**12.1** from Coleman]

   When we attempted to quantize the free Dirac theory

   $$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - m)\psi$$

   with canonical commutation relations, we encounter a disastrous contradiction with the positivity of the energy. We succeeded when we used canonical anticommutators. Much earlier we were able to quantize the free charged scalar field,

   $$\mathcal{L} = \left(\partial_\mu \phi^* \partial^\mu \phi - \mu^2 \phi^* \phi\right)$$

   with canonical commutators. Attempt to quantize the free charged scalar field with (nearly) canonical anticommutators:

   $$\begin{aligned} \{\phi(\boldsymbol{x},t), \phi(\boldsymbol{y},t)\} &= \{\dot{\phi}(\boldsymbol{x},t), \dot{\phi}(\boldsymbol{y},t)\} = 0 \\ \{\phi(\boldsymbol{x},t), \phi^*(\boldsymbol{y},t)\} &= \{\dot{\phi}(\boldsymbol{x},t), \dot{\phi}^*(\boldsymbol{y},t)\} = 0 \\ \{\phi(\boldsymbol{x},t), \dot{\phi}^*(\boldsymbol{y},t)\} &= \lambda \delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \end{aligned}$$

   where λ is a (possibly complex) constant. Show that one reaches a contradiction with the positivity of the norm in Hilbert space; that is to say

   $$\langle \phi | \{\theta, \theta^\dagger\} | \phi \rangle \geq 0$$

   for any operator θ and any state $|\phi\rangle$.

   Hints:

   (a) Canonical anticommutation implies that, even on the classical level, ϕ and ϕ^* are Grassman variables. If you don't take proper account of this (especially in ordering terms when deriving the canonical momenta), you'll get hopelessly confused.

   (b) Dirac theory is successfully quantized using anticommutators; the sign of the Lagrangian is fixed by appealing to the positivity of the inner product in Hilbert space. If we attempt to quantize the theory using commutators, we get into trouble with the positivity of the energy. The Klein-Gordon theory is successfully quantized using commutators. So it's to be expected that we'd get into trouble, if we attempted to quantize the Klein-Gordon theory with anticommutators, with the positivity of the inner product.

2. [**3.5** from Peskin & Schröder]

It is possible to write field theories with continuous symmetries linking fermions and bosons; such transformations are called supersymmetries.

(a) The simplest example of a supersymmetric field theory is the theory of a free complex boson and a free Weyl fermion, written in the form

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + \chi^\dagger i \overline{\sigma} \cdot \partial \chi + F^* F.$$

Here F is an auxiliary complex scalar field whose field equation is $F = 0$. Show that this Lagrangian is invariant (up to a total divergence) under the infinitesimal tranformation

$$\begin{aligned} \delta\phi &= -i\epsilon^T \sigma^2 \chi, \\ \delta\chi &= \epsilon F + \sigma \cdot \partial\phi \sigma^2 \epsilon^*, \\ \delta F &= -i\epsilon^\dagger \overline{\sigma} \cdot \partial\chi, \end{aligned}$$

where the parameter ϵ_a is a 2-component spinor of Grassmann numbers, and σ^2 the second Pauli matrix.

(b) Show that the term

$$\Delta\mathcal{L} = \left[m\phi F + \tfrac{1}{2} i m \chi^T \sigma^2 \chi \right] + \text{(complex conjugate)}$$

is also left invariant by the transformation given in part (a). Eliminate F from the complete Lagrangian $\mathcal{L} + \Delta\mathcal{L}$ by solving its field equation, and show that the fermion and boson fields ϕ and χ are given the same mass.