Quantum Field Theory II FS24

Series 1 - Perturbation theory: Divergences and CTs 21.02.2024

In the first lecture, you saw that for the so-called Model 2 (Coleman's book), with Hamiltonian density

$$\mathcal{H}_I = g\phi(x)\rho(\mathbf{x}), \tag{1}$$

where $\rho(\mathbf{x})$ is time-independent, one gets an absurd T-dependent result for $\langle 0| S |0\rangle$. You also saw why this problem arises and how it is tackled with the introduction of a counterterm (CT) in the Hamiltonian density, that you tune so that you get $\langle 0| S |0\rangle = 1$. This is a first introduction to the idea of CTs and their significance.

Now let's try to compute the S matrix for Model 2. We work with the Hamiltonian with the counterterm

$$H_I = [g \int d^3\mathbf{x}\phi(x)\rho(\mathbf{x}) - a]f(t), \tag{2}$$

with $f(t)$ the usual adiabatic function.

1. How many connected Wick diagrams one has in this case and why ?

Hint: See also discussion of Model 1 in Coleman, Ch. 8.

2. Draw the diagrams, define the operators corresponding to these diagrams and write the expression for the S matrix in this case.

3. Only two of these contributions contribute to the vacuum-to-vacuum matrix element $\langle 0| S |0\rangle$. Which are these and why should their sum be zero ?

Therefore, only one operator contributes to S.

4. Calculate the contribution to the S matrix.

5. Argue why $\lim_{T\to\infty} S = 1$ from the result of 4.

Now let's try to compute the ground-state energy for the model 2. The condition for the cancellation of the two diagrams (call them D_2 and D_3) is

$$\lim_{T\to\infty} \left[\frac{\mathcal{O}_2}{2!} + \mathcal{O}_3\right] = 0 \tag{3}$$

with $\mathcal{O}_2$ and $\mathcal{O}_3$ the corresponcing operators we have calculated in 2.

6. Show that

$$\frac{\mathcal{O}_2}{2!} = \frac{ig^2}{2!} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{|\tilde{\rho}(\mathbf{p})|^2}{|\mathbf{p}|^2 + \mu^2} \int \frac{d\omega_{\mathbf{p}}}{2\pi} |\tilde{f}(\omega_{\mathbf{p}})|^2 \tag{4}$$

with the quantities with the tilde representing the Fourier tranform of $\rho(\mathbf{x})$ and $f(t)$ respectively. *Hint: Think of the behaviour of $\tilde{f}(\omega_{\mathbf{p}})$ in the limit $T \to \infty$.*

7. Calculate the ground energy E_0 of the theory.

*Hint: Use **Parseval's Theorem**, which states that*

$$\int \frac{d\omega_{\mathbf{p}}}{2\pi} |\tilde{f}(\omega_{\mathbf{p}})|^2 = \int dt |f(t)|^2 \tag{5}$$

and the relation between $\mathcal{O}_2$ and $\mathcal{O}_3$ from above.

8. Write now the expression for E_0 into the position space and define

$$V(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{|\mathbf{p}|^2 + \mu^2} \,. \tag{6}$$

What does this form remind you from electrostatics ? Observe some similarities and differences between the two cases.

9. Perform this integral and find an expression for $V(\mathbf{x})$. This potential is called **Yukawa potential**.

Hint: Define $|\mathbf{p}| = p$, $|\mathbf{x}| = r$ and move to spherical coordinates and reduce the integral into an integral over p. Then use Cauchy's theorem to compute the final integral.

10. Observe that if $\rho(\mathbf{x}) \to \delta^3(\mathbf{x})$, then $E_0 \to \infty$.

This divergence is called an **ultraviolet divergence**, because in $\mathbf{p}$-space it corresponds to the integral blowing up at high $|\mathbf{p}|$. We will see how to deal with these later on in the course.

Now let's calculate

$$\langle \mathbf{p}_1, ..., \mathbf{p}_n| \, |0\rangle_P \,. \tag{7}$$

with $|0\rangle_P$ the physical vacuum and $|\mathbf{p}_1, ..., \mathbf{p}_n\rangle$ eigenstates of the non-interacting Hamiltonian H_0. Consider that $f(t) = e^{\epsilon t}$, for $t < 0$ and $f(t) = 0$, for $t > 0$.

11. Argue that $U_I(-\infty, +\infty) \, |0\rangle = |0\rangle_P$.

12. Use the results from Model 1 (Coleman, Ch. 8) to prove that:

$$\langle \mathbf{p}_1, ..., \mathbf{p}_n| \, U_I(-\infty, +\infty) \, |0\rangle = e^{\frac{1}{2}(-\alpha + i\beta)} h^*(\mathbf{p}_1) h^*(\mathbf{p}_1) \cdots h^*(\mathbf{p}_n), \tag{8}$$

with

$$h(\mathbf{p}) = \frac{-ig\tilde{\rho}(\mathbf{p})\tilde{f}(\omega_{\mathbf{p}})}{(2\pi)^{3/2}\sqrt{2\omega_{\mathbf{p}}}} \tag{9}$$

13. Calculate $\alpha = \int d^3\mathbf{p} |h(\mathbf{p})|^2$ in the limit $\epsilon \to 0$.

Hint: Calculate first the Fourier transform of $f(t)$ in the limit $\epsilon \to 0$.

14. If $\rho(\mathbf{x}) \to \delta^3(\mathbf{x})$, what is the behaviour of α ? Is it divergent and, if yes, how does it diverge ? Is it equally bad as for the divergence for the case of E_0 in 10. ?

15. Prove that $\lim_{\mu\to 0} \alpha = \infty$. This is called an **infrared divergence**.

16. Prove that $\lim_{\mu\to 0} < E >$ is finite, with

$$< E >= \int d^3\mathbf{p} |h(\mathbf{p})|^2 \omega_{\mathbf{p}}. \tag{10}$$

This means that this divergence is also unphysical in the sense that, although an infinite number of photons are radiated in this process, only a finite amount of energy is radiated.