Quantum Field Theory II — FS24

Series 3 - Meson self-energy at $\mathcal{O}(g^2)$ and one-loop integrals 13.03.2024

The goal of the first part of this series is to calculate in two different ways the one-loop integral of the following form:

$$I_n = \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2 + a + i\epsilon)^n} \tag{1}$$

1. Write the integral explicitly in terms of $q = (q_0, \mathbf{q})$ and do the plot of the poles in the complex q_0 plane.

2. Perform the **Wick rotation** $q_0 = iq_4$ to translate the integral from Minkowski to Euclidean space and prove that the integral becomes

$$I_n = i \int \frac{d^4q_E}{(2\pi)^4} \frac{1}{(-q_E^2 + a + i\epsilon)^n} \tag{2}$$

with $q_E = (q_4, \mathbf{q})$.

We now have a four-dimensional, spherically symmetric integral to do in Euclidean space.

3. Argue that

$$\int d^4q_E f(q_E^2) = \alpha \int_0^\infty dz\ z f(z) \tag{3}$$

with α some constant arising from the angular integration.

4. Prove that $\alpha = \pi^2$.

Hint: You can determine α without having to go to spherical coordinates in four-dimensional space, since α is a universal constant. Therefore, you can calculate it by considering any function f (consider $f = e^{-q_E^2}$). Remember that $\int dx\ e^{-x^2} = \sqrt{\pi}$.

5. Calculate I_1 considering the integration up to a specific upper limit (cut-off) Λ with $\Lambda \gg a$. From this calculate also I_2.

Hint: To calculate I_n, you only need to I_1 and you can get the rest by differentiating I_1 with respect to a.

Now we will calculate the same quantity in the context of the **dimensional regularization**.

Let's calculate a Wick-rotated Feynman integral of the type

$$\int \frac{d^dq_E}{(2\pi)^d} \frac{1}{(q_E^2 + a)^2} = \int \frac{d\Omega_d}{(2\pi)^d} \int_0^\infty dq_E \frac{q_E^{d-1}}{(q_E^2 + a)^2} \tag{4}$$

over a d-dimensional Euclidean space.

6. Prove that

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{5}$$

Hint: Start from the Euclidean integral $\int dx\ e^{-x^2} = \sqrt{\pi}$ *and raise it to the power d. Reinterpret the integral as d-dimensional Gaussian integral which you can solve by splitting the angular and the radial integration. The angular one is the one you need to calculate. For the radial one use the definition of the Gamma function.*

$$\Gamma(z) = \int_0^\infty dt\ t^{z-1} e^{-t}. \tag{6}$$

7. Prove that

$$\int \frac{d^d q_E}{(2\pi)^d} \frac{1}{(q_E^2 + a)^2} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(2 - d/2)}{\Gamma(2)} \Big(\frac{1}{a}\Big)^{2-d/2}. \tag{7}$$

Hint: Use the definition of the Beta function

$$\int_0^1 dx\ x^{\alpha-1}(1-x)^{\beta-1} = B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{8}$$

8. Since $\Gamma(z)$ has isolated poles at $z = 0, -1, -2, ...$ this integral has poles at $d = 4, 6, 8, ...$ Take the limit $\epsilon \equiv 4 - d \to 0$ and show that in this limit

$$\int \frac{d^d q_E}{(2\pi)^d} \frac{1}{(q_E^2 + a)^2} \to (4\pi^2)\Big(\frac{2}{\epsilon} - \log(a) - \gamma + \log(4\pi) + \mathcal{O}(\epsilon)\Big). \tag{9}$$

with $\gamma \approx 0.5772$ the Euler-Mascheroni constant.

Hint: Consider the expansion $\Gamma(2 - d/2) = \Gamma(\epsilon/2) = \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)$.

9. Compare this result with the expression for I_2 from question 5. What does the $1/\epsilon$ pole correspond to ?