# Quantum Field Theory II FS24

**Series 4 - Further important calculations on renormalization** 20.03.2024

**Part 1:** In the lectures, you arrived at the following integral expression for $\tilde{\Pi}'$

$$\tilde{\Pi}'(p^2) = \frac{g^2}{16\pi^2}\int_0^1 dx\Big[\ln\Big(\frac{m^2 - p^2 x(1-x) - i\epsilon}{m^2 - \mu^2 x(1-x)}\Big) + (p^2-\mu^2)\frac{x(1-x)}{m^2-\mu^2 x(1-x)}\Big]. \quad (1)$$

Now, let's further investigate this expression and derive some interesting results.

1. Compute the imaginary part of the expression above to $\mathcal{O}(g^2)$.

*Hint: Follow the discussion in Coleman's paragraph 16.1. Consider also that Im* $\ln(-\xi - i\epsilon) = -\pi\theta(\xi)$, *for* $\xi$ *real.*

2. Use the formula for $\tilde{D}'(p^2)$ and its spectral representation to show that, to all orders,

$$Im\ \tilde{\Pi}'(p^2) \propto |\tilde{D}'(p^2)|^{-2}\sigma(p^2). \quad (2)$$

What can be said about the case $p^2 \to \mu^2$ ?

*Hint: Use the following formula*

$$Im\Big[\frac{1}{p^2-\mu^2+i\epsilon}\Big] = -\pi\delta(p^2-\mu^2). \quad (3)$$

*and consider separately the cases* $p \neq \mu^2$ *and* $p = \mu^2$ *(the second case as a limit).*

3. This question can be broken down to three parts:

3a. Prove that for $\langle n|\,\phi'(y)\,|0\rangle$, with $|n\rangle$ to be an out state $|k_1,...,k_n\rangle$, one gets

$$\langle k_1,...,k_n|\,\phi'(y)\,|0\rangle = \int \frac{d^4q}{(2\pi)^4}e^{iq\cdot y}\frac{k_1^2-\mu^2}{i}\cdots\frac{k_n^2-\mu^2}{i}\tilde{G}'^{(n+1)}(-k_1,...,-k_n,q). \quad (4)$$

*Hint: Use equations (14.15) and (14.37) from Coleman to express this matrix element in terms of an integral over an appropriate renormalized Green's function.*

3b. Argue why there is exactly one contribution to exactly one Green's functions $\tilde{G}'^{(3)}(-k,-k',q)$ for the Model 3 to $\mathcal{O}(g)$. Draw the respective Feynman diagram for this Green's function and write its mathematical expression.

3c. Prove that $\sigma(p^2)$ to $\mathcal{O}(g^2)$ is

$$\sigma(p^2) = \Big|\frac{g}{p^2-\mu^2}\Big|^2\frac{1}{2\pi}\int\frac{d^3\mathbf{k}}{(2\pi)^3 2E_k}\frac{d^3\mathbf{k}'}{(2\pi)^3 2E_{k'}}(2\pi)^4\delta^{(4)}(p-k-k'). \quad (5)$$

*Hint: Remember the formula for the spectral density* $\sigma(p^2)$ *in terms of the matrix element of a renormalized field between the vacuum and a multiparticle state (equation (15.12) from Coleman).*

3d. Verify that the answer in 3c. is consistent with the answer in 2. combined with 1.

*Hint: Integrate the expression in 3c. and work in the center-of-momentum frame (use the expression (12.24) from Coleman).*

**<u>Part 2:</u>** Now, we turn our attention to a different topic and to another very important calculation.

4. Calculate the vertex of Model 3, $-i\tilde{\Gamma}'(p^2, p'^2, q^2)$ to $\mathcal{O}(g^3)$, as an integral over (two) Feynman parameters, for $p^2 = p'^2 = m^2$.

*Hint: You may find the following formula useful*

$$\int \frac{d^4k'}{(2\pi)^4} \frac{1}{(k'^2 + a)^3} = \frac{i}{32\pi^2 a}. \tag{6}$$

5. Show that this is an analytic function of $q^2$ in the entire complex $q^2$ plane except for a cut along the positive real axis beginning at $q^2 = 4m^2$ (This function also has interesting analytic properties when all three arguments are complex, but untangling the analytic structure of a function of three complex variables is a bit too much work for a homework problem).

*Hint: You may find it helpful - but definitely not necessary - to think in* $(w, z)$ *instead of* $(x, y)$ *via the following substitution:* $w = x + y$, $z = x - y$, *with* $w \in [0, 1]$ *and* $z \in [-w, w]$, *and* $x$, $y$ *the integration variables.*