Quantum Field Theory II FS24

Series 5 - Unstable particles 27.03.2024

The goal of this exercise is to calculate the exponential decay law for unstable particles.

As in the case of Breit-Wigner formula, we will use

$$\mathcal{L} \to \mathcal{L} + f(x)\phi(x). \tag{1}$$

The amplitude for detecting the particle at a distance y away from the spacetime point where the particle is produced is

$$A(y) = \int d^4x' d^4x f(x'-y)^* f(x) \langle 0| T(\phi(x')\phi(x)) |0\rangle , \tag{2}$$

assuming that y is far later in time than x (hence the time ordering).

1. Prove that

$$A(y) = \int \frac{d^4k}{(2\pi)^4} |\tilde{f}(k)|^2 e^{-ik\cdot y} \frac{i}{k^2 - \mu^2 + i\mu\Gamma} . \tag{3}$$

Hint: Assume that $f(x)$ is real, so that $\tilde{f}(k) = \tilde{f}(-k)^*$, and also that for the propagator one has the following approximation

$$\tilde{D}'(k^2) = \frac{i}{k^2 - \mu^2 + i\mu\Gamma}. \tag{4}$$

Now, we want to calculate analytically the expression above for large y^2. We will do that via the **method of stationary phase**, so let's open a small parenthesis to describe this method.

Suppose we have an integral of the form

$$I = \int dt e^{i\theta(t)} g(t), \tag{5}$$

with $\theta(t)$ a real function that varies rapidly compared to the rate at which $g(t)$ varies. The main contribution to this integral comes at points of *stationary phase*, where

$$\frac{d\theta(t)}{dt} = 0, \tag{6}$$

because at these points $\theta(t)$ is not varying at all. Let's assume in our case that there is one such point t_0. We therefore approximate the integral by its value near the stationary phase point. Then, we get

$$I = e^{i\theta(t_0)} g(t_0) \int dt e^{\frac{1}{2}\theta''(t_0)(t-t_0)^2}. \tag{7}$$

This is a trivial Gaussian integral and the result is

$$I = e^{i\theta(t_0)} g(t_0) \sqrt{\frac{2\pi}{|\theta''(t_0)|}} e^{i(\pi/4)sgn(\theta''(t_0))}. \tag{8}$$

Our integral in (3) is of stationary phase form, because we have a complex exponential with argument $k \cdot y$, and all four components of y are large. So we have four integrals we can do by stationary phase.

2. Use the method of stationary phase to prove that

$$A(y) = e^{i\pi} \frac{\mu}{2(2\pi)^2} |\tilde{f}(k_0)|^2 \int_0^\infty ds\ e^{-\frac{\Gamma s}{2}}\ \frac{1}{s^2}\ e^{-i\theta(s)}, \tag{9}$$

with $\theta(s) = \dfrac{\mu y^2}{2s} + \dfrac{\mu s}{2}$.

Hint: Write the propagator as an integral using the following relation

$$\frac{i}{k^2 - \mu^2 + i\mu\Gamma} = \int_0^\infty \frac{ds}{2\mu} e^{i(s/2\mu)(k^2-\mu^2+i\mu\Gamma)}. \tag{10}$$

3. Use again the method of stationary phase to perform this last integral and prove that the final result is

$$A(y) = -\sqrt{\frac{\mu}{32\pi^3}} e^{i\pi/4} |\tilde{f}(k_0)|^2 e^{-\frac{\Gamma s_0}{2}}\ s_0^{-3/2}\ e^{-i\mu s_0}, \tag{11}$$

with $k_0 = \mu(y/s_0)$ and $s_0 = \sqrt{y^2}$.

Hint: Since (9) *has the large factor* y^2 *in it, one can use the method of stationary phase for this integral too.*